# Power in AC Circuits

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*The LORD is my strength and song, and is become my salvation.* Psalm 118:14

Power analysis of linear AC circuits with sinusoidal excitation is a standard topic of introductory electric-circuits textbooks. This document is meant as a supplement to a textbook chapter on power. Its goal is to provide additional insight into the meaning of the various AC power concepts by means of a time-domain analysis. The relationship between the various power concepts is explained and the main results are summarized.

**Instantaneous Power.** Consider a two-terminal network or component, as shown in Figure 1(a). The absorbed instantaneous power is defined as

$$p(t) = v(t)i(t). \tag{1}$$

The instantaneous power can also be calculated for a network or component with more than two terminals. Consider the component with $n$ terminals shown in Figure 1(b). Let $v_1$, $v_2$, ..., $v_n$ be the voltages of the $n$ terminals. Suppose that $v_1$, $v_2$, ..., $v_n$ are defined with respect to a common reference point, where the reference point is marked by the ground symbol in the figure. The instantaneous power has the formula

$$p(t) = \sum_{k=1}^{n} v_k(t)i_k(t). \tag{2}$$

Note that if the reference point is changed, the voltages $v_1$, $v_2$, ..., $v_n$ are changed by the same amount. If the voltage of the new reference point is $v_0$ with respect to the old reference point, then $v_1 \to v_1 + v_0$, $v_2 \to v_2 + v_0$, ..., $v_n \to v_n + v_0$. However, the result of formula (2) will not change, since Kirchhoff's current law implies that $v_0 \sum i_k = 0$. It could be noted also that equation (1) is the same as formula (2) when the reference point is the terminal where $v(t)$ has the $-$ sign.

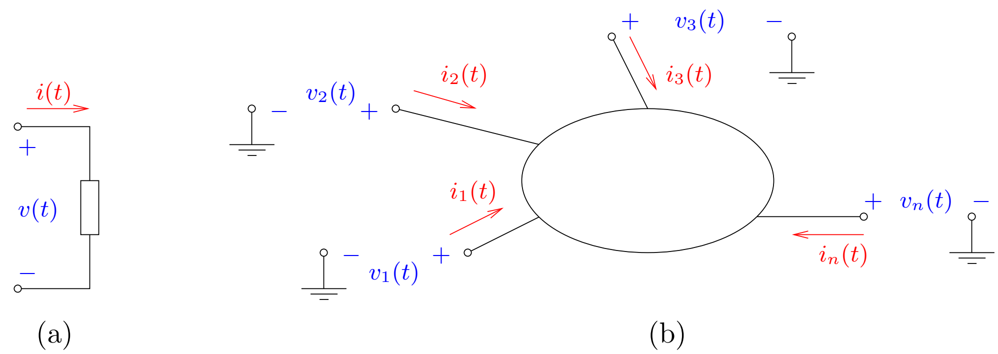



Figure 1: Components with two or more terminals.

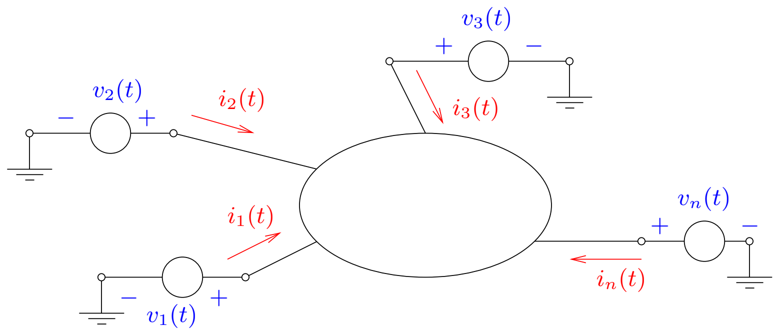



Figure 2: An equivalent representation of the $n$-terminal component of Figure 1(b).

Formula (2) can be justified by considering the circuit of Figure 2, showing an equivalant representation of the $n$-terminal component of Figure 1(b). In this equivalent representation, the currents and voltages of the component are caused by $n$ sources of current or voltage. The power absorbed by the component is the same, since the component has the same voltages and currents as in Figure 1(b). Now the power generated by the $n$ sources must equal the power $p(t)$ absorbed by the component. Then, equation (2) is obtained based on the observation that the total power generated by the sources is $\sum_{k=1}^{n} v_k(t) i_k(t)$.

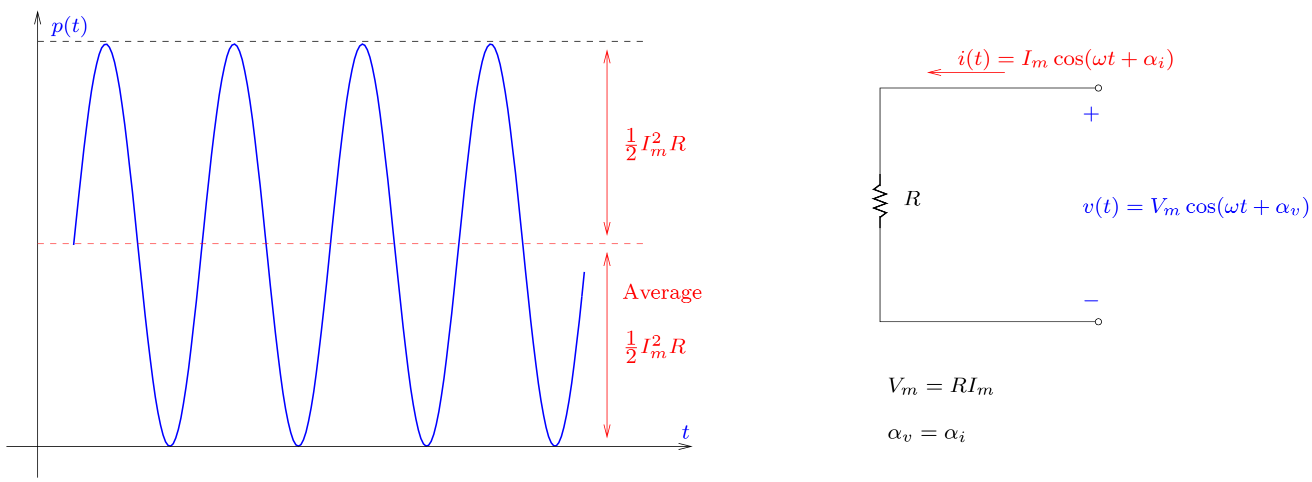



Figure 3: The instantaneous power of a resistor.

**Examples.** The application of equation (1) to AC circuits will be illustrated on a few examples. First, assume that the two-terminal component of Figure 1(a) is a resistor of value $R$. In view of Ohm's law, $p(t) = Ri(t)^2$. Moreover, if $i(t) = I_m \cos(\omega t + \alpha_i)$, then

$$p(t) = RI_m^2 \cos(\omega t + \alpha_i)^2.$$

Recalling the formula $\cos(x)^2 = \dfrac{1 + \cos(2x)}{2}$, we obtain

$$p(t) = \frac{RI_m^2}{2} + \frac{RI_m^2}{2} \cos(2\omega t + 2\alpha_i). \tag{3}$$

The equation indicates that the instantaneous power oscillates between 0 and $RI_m^2$, with an average of $\frac{RI_m^2}{2}$, as shown also in Figure 3.

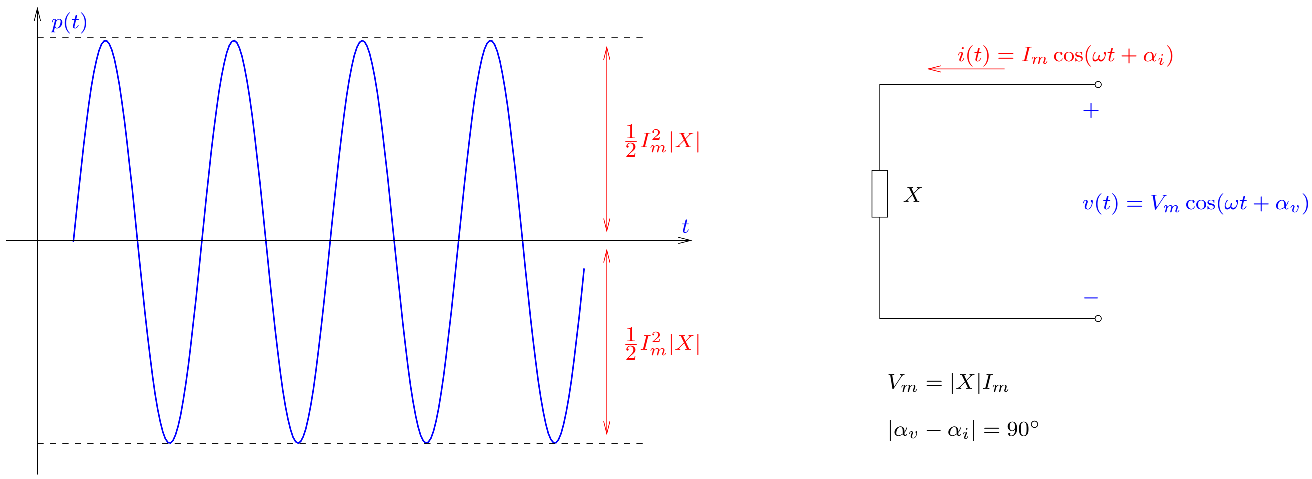



Figure 4: The instantaneous power of a reactive component.

Assume now that the two-terminal component of Figure 1(a) is a capacitor of value $C$. Since $i(t) = C\dot{v}(t)$, assuming $v(t) = V_m\cos(\omega t + \alpha_v)$,

$$p(t) = -C\omega V_m^2\cos(\omega t + \alpha_v)\sin(\omega t + \alpha_v).$$

Recalling that $\cos(x)\sin(x) = \frac{\sin(2x)}{2}$,

$$p(t) = -\frac{1}{2}C\omega V_m^2\sin(2\omega t + 2\alpha_v). \tag{4}$$

The equation indicates that the instantaneous power oscillates between $-\frac{C\omega V_m^2}{2}$ and $\frac{C\omega V_m^2}{2}$ with a zero average. Recalling that the reactance $X = -\frac{1}{C\omega}$ and $V_m = |X|I_m$, we could also say that the instantaneous power oscillates with an amplitude of $\frac{1}{2}I_m^2|X|$, as shown in Figure 4.

In the same way as before, assuming that the two-terminal component of Figure 1(a) is an inductor of value $L$,

$$p(t) = -\frac{1}{2}L\omega I_m^2\sin(2\omega t + 2\alpha_i). \tag{5}$$

The graph of $p(t)$ is shown in Figure 4. As could be seen from equations (4) and (5). the average power of a reactive component (a capacitor or an inductor) is zero.

**Average, Apparent, and Reactive Powers.** Substituting the voltage $v(t) = V_m\cos(\omega t + \alpha_v)$ and the current $i(t) = I_m\cos(\omega t + \alpha_i)$ in equation (1):

$$p(t) = V_m I_m\cos(\omega t + \alpha_v)\cos(\omega t + \alpha_i). \tag{6}$$

Recalling the formula $\cos(x)\cos(y) = \frac{\cos(x-y) + \cos(x+y)}{2}$,

$$p(t) = \frac{V_m I_m}{2}\cos(\alpha_v - \alpha_i) + \frac{V_m I_m}{2}\cos(2\omega t + \alpha_i + \alpha_v). \tag{7}$$

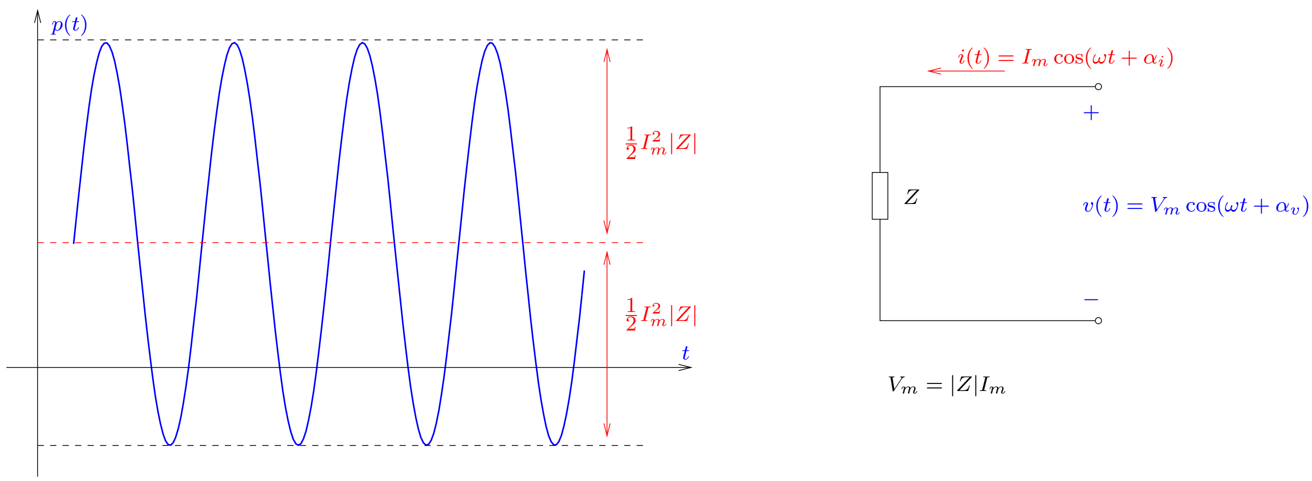



Figure 5: The instantaneous power for a component or network of impedance $Z$.

The graph of $p(t)$ is shown in Figure 5.

Considering equation (7), the constant term is the average of $p(t)$. Thus, the average power is

$$P = \frac{V_m I_m}{2} \cos(\alpha_v - \alpha_i). \tag{8}$$

By definition, the apparent power is

$$S = \frac{V_m I_m}{2} \tag{9}$$

In view of equation (7), the instantaneous power can be written as

$$p(t) = P + S\cos(2\omega t + \alpha_i + \alpha_v). \tag{10}$$

Applying the formula $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ to the above equation with $x = \alpha_v - \alpha_i$ and $y = 2\omega t + 2\alpha_i$, the instantaneous power can be written in the form

$$p(t) = P + S\cos(\alpha_v - \alpha_i)\cos(2\omega t + 2\alpha_i) - S\sin(\alpha_v - \alpha_i)\sin(2\omega t + 2\alpha_i). \tag{11}$$

In view of equations (8) and (9),

$$P = S\cos(\alpha_v - \alpha_i). \tag{12}$$

Moreover, the amplitude of the last term of equation (11) is known as reactive power:

$$Q = S\sin(\alpha_v - \alpha_i). \tag{13}$$

Equation (11) can be written in terms of $P$ and $Q$ as

$$p(t) = P + P\cos(2\omega t + 2\alpha_i) - Q\sin(2\omega t + 2\alpha_i). \tag{14}$$

**Physical Interpretation.** Equation (14) has a simple physical interpretation. Assume a load represented by a resistor of value $R$ in series with a reactive component of reactance $X$. The instantaneous power of the load can be decomposed into the sum of the power of the resistor $p_R(t)$ and the power of the reactive component $p_X(t)$:

$$p(t) = p_R(t) + p_X(t). \tag{15}$$

In view of equations (3), (4), and (5), $p_R(t)$ and $p_X(t)$ can be identified with the terms of equation (14) as follows:

$$p_R(t) = P + P\cos(2\omega t + 2\alpha_i) \quad (16)$$

$$p_X(t) = -Q\sin(2\omega t + 2\alpha_i). \quad (17)$$

We can conclude that the reactive power $Q$ is the peak rate at which energy flows to or from the reactive component. We arrive to the same conclusion if the load is a parallel combination of a resistor of conductance $G$ and a reactive component of susceptance $B$. First, let us note that if we apply to equation (10) the formula $\cos(y - x) = \cos(x)\cos(y) + \sin(x)\sin(y)$ with $x = \alpha_v - \alpha_i$ and $y = 2\omega t + 2\alpha_v$, we obtain:

$$p(t) = P + P\cos(2\omega t + 2\alpha_v) + Q\sin(2\omega t + 2\alpha_v). \quad (18)$$

Denoting by $p_G(t)$ and $p_B(t)$ the instantaneous powers of the resistor and the reactive component, respectively,

$$p(t) = p_G(t) + p_B(t). \quad (19)$$

In view of equations (3), (4), and (5), $p_G(t)$ and $p_B(t)$ can be identified with the terms of equation (18) as follows:

$$p_G(t) = P + P\cos(2\omega t + 2\alpha_v) \quad (20)$$

$$p_B(t) = Q\sin(2\omega t + 2\alpha_v). \quad (21)$$

**More on the Reactive Power.** The average power $P$ was defined as the average of the instantaneous power $p(t)$. The reactive power also is the average of a function $q(t)$ defined as follows. Given the $n$-terminal component of Figure 1(b), let

$$q(t) = \sum_{k=1}^{n} v_k(t) i_k(t + t_0), \quad (22)$$

where $t_0 = \frac{\pi}{2\omega}$. It has been assumed here that all currents and voltages are sinusoidal and have the frequency $\omega$. Note that in the case of the two-terminal component of Figure 1(a), the equation can be simplified to

$$q(t) = v(t)i(t + t_0). \quad (23)$$

Substituting $v(t) = V_m\cos(\omega t + \alpha_v)$ and $i(t) = I_m\cos(\omega t + \alpha_i)$, noticing also that $i(t + t_0) = -I_m\sin(\omega t + \alpha_i)$, we obtain

$$q(t) = -V_m I_m \cos(\omega t + \alpha_v)\sin(\omega t + \alpha_i). \quad (24)$$

Recalling that $\cos(x)\sin(y) = \dfrac{-\sin(x - y) + \sin(x + y)}{2}$,

$$q(t) = \frac{1}{2}V_m I_m \sin(\alpha_v - \alpha_i) - \frac{1}{2}V_m I_m \sin(2\omega t + \alpha_v + \alpha_i), \quad (25)$$

that is,

$$q(t) = Q - S\sin(2\omega t + \alpha_v + \alpha_i). \quad (26)$$

The equation above shows that $Q$ is the average of $q(t)$.

**Additivity of Power.** It seems very intuitive that the instantaneous power of a network should be the sum of the instantaneous powers of each network component. This, however, can also be proven mathematically. In fact, a more general result can be proven. It can be shown that given a power-like function of voltage and current (it will be defined precisely next), its value for a network is the sum of its values for each network component.

Consider a network with $n$ terminals. Let $v_1, v_2, \dots v_n$ be the terminal voltages with respect to the reference. Let $i_1, i_2, \dots i_n$ be the currents flowing into the network through the terminals $1, 2, \dots, n$. Let:

$$u(t, t') = \sum_{k=1}^{n} v_k(t) i_k(t'). \tag{27}$$

Note that if $t = t'$, then $u$ is the instantaneous power of the network $p(t)$. However, as defined, $u$ is a power-like function that may not have a physical semnification. Suppose that the network has $N$ nodes. Without loss of generality, we will assume that there is no node connected to more than one terminal and that each terminal is connected to some node (so $N \geq n$.) Using the indices $1, 2, \dots, n$ for the nodes connected to a terminal and $n+1, n+2, \dots, N$ for the nodes that are not connected to a terminal, let $i_k = 0$ for all $k > n$. It follows that

$$\sum_{k=1}^{n} v_k(t) i_k(t') = \sum_{k=1}^{N} v_k(t) i_k(t'). \tag{28}$$

Let $i_{k,c}$ be the current flowing from the node $k$ to the component $c$, for $k = 1, 2, \dots, N$ and $c = 1, 2, \dots, m$. If the component $c$ is not connected to the node $k$, then $i_{k,c} = 0$. Let $u_c$ be defined as in equation (27) for each component $c$:

$$u_c(t, t') = \sum_{k=1}^{N} v_k(t) i_{k,c}(t'). \tag{29}$$

By Kirchhoff's current law,

$$\sum_{k=1}^{N} v_k(t) i_k(t') = \sum_{k=1}^{N} \left( v_k(t) \sum_{c=1}^{m} i_{k,c}(t') \right). \tag{30}$$

By changing the summation order,

$$\sum_{k=1}^{N} v_k(t) i_k(t') = \sum_{c=1}^{m} \sum_{k=1}^{N} v_k(t) i_{k,c}(t'), \tag{31}$$

that is,

$$u(t, t') = \sum_{c=1}^{m} u_c(t, t'). \tag{32}$$

**Total Average Power and Total Reactive Power.** In view of equations (2) and (27), the instantaneous power satisfies $p(t) = u(t, t)$. Additionally, using equation (29), let $p_c(t) = u_c(t, t)$ be the instantaneous power on the component $c$. Applying equation (32), the instantaneous power of the network can be calculated by adding the instantaneous powers of its components:

$$p(t) = \sum_{c=1}^{m} p_c(t). \tag{33}$$

Therefore, the average power of the network is

$$P = \sum_{c=1}^{m} P_c, \tag{34}$$

where $P_c$ is the average power of the component $c$. In view of equations (22) and (27), $q(t) = u(t, t + t_0)$, where $t_0 = \frac{\pi}{2\omega}$. Additionally, using equation (29), let $q_c(t) = u_c(t, t + t_0)$. Applying equation (32):

$$q(t) = \sum_{c=1}^{m} q_c(t). \tag{35}$$

It follows that the reactive power of the network is

$$Q = \sum_{c=1}^{m} Q_c, \tag{36}$$

where $Q_c$ is the average of $q_c(t)$, that is, the reactive power of the component $c$.

**Relation of $p(t)$ to $P$, $Q$, and $S$.** Given an arbitrary network with two terminals, Figure 6 illustrates the relation between the instantaneous power $p(t)$ and the apparent power $S$, the average power $P$, and the reactive power $Q$. The figure considers also the special cases when the network is equivalent with a resistor or with a reactive component. The following remarks could be made:

- The peak power absorbed by the load is $S + P$.
- The maximum rate at which energy is returned to the source is $S - P$. Note that $0 \leq S - P \leq |Q|$. While the maximum rate at which the reactive components return energy is $|Q|$, part of this energy is used by the resistive components, and thus not all of it is sent back to the source.

Note Table 1 for examples illustrating the calculation of the average and reactive powers.

For a network with $n > 2$ terminals, the remarks above apply at each terminal $k$ for $p_k(t)$, $P_k$, $Q_k$, and $S_k$, where $p_k(t) = v_k(t)i_k(t)$, $v_k(t) = V_{mk}\cos(\omega t + \alpha_{vk})$, $i_k(t) = I_{mk}\cos(\omega t + \alpha_{ik})$, $P_k = \frac{V_{mk}I_{mk}}{2}\cos(\alpha_{vk} - \alpha_{ik})$, $Q_k = \frac{V_{mk}I_{mk}}{2}\sin(\alpha_{vk} - \alpha_{ik})$, and $S_k = \frac{V_{mk}I_{mk}}{2}$. To minimize losses, power factor correction could be carried out for each of the $n$ terminals separately. As for the overall powers $p(t)$, $P$, $Q$, and $S$, if $S$ is defined as $S = \sqrt{P^2 + Q^2}$, then in general $S + P$ and $S - P$ will not be the peak value of $p(t)$ and the minimum value of $p(t)$, respectively. Moreover, in general $|Q|$ will not be the maximum rate at which reactive components return energy. The properties stated above for $p(t)$, $P$, $Q$, and $S$ in the case $n = 2$ are not satisfied for $n > 2$ because the phase angle difference $\alpha_{vk} - \alpha_{ik}$ does not have to be the same at each terminal $k$.

**The Power Factor.** Considering a load with two terminals, using the notation of Figure 1(a) with $v(t) = V_m\cos(\omega t + \alpha_v)$ and $i(t) = I_m\cos(\omega t + \alpha_i)$, the power factor is defined as $PF = \cos(\alpha_v - \alpha_i)$. When $PF = 1$, the load is equivalent to a resistor and energy flows always from the source to the load; the load does not return any energy back to the source. When $PF < 1$, in every cycle of $p(t)$, the load takes more energy than it needs and returns the excess of energy back to the source. Ideally, this would not be a problem. In practice, however, energy cannot be transferred without losses. Therefore, it is best if the load takes exactly as much energy as it needs, without returning any back to the source. For this reason, power factor correction is used to ensure that the load operates at a power factor $PF = 1$. The $p(t)$ curves with the same average power $P$ but different power factors are illustrated in Figure 7.

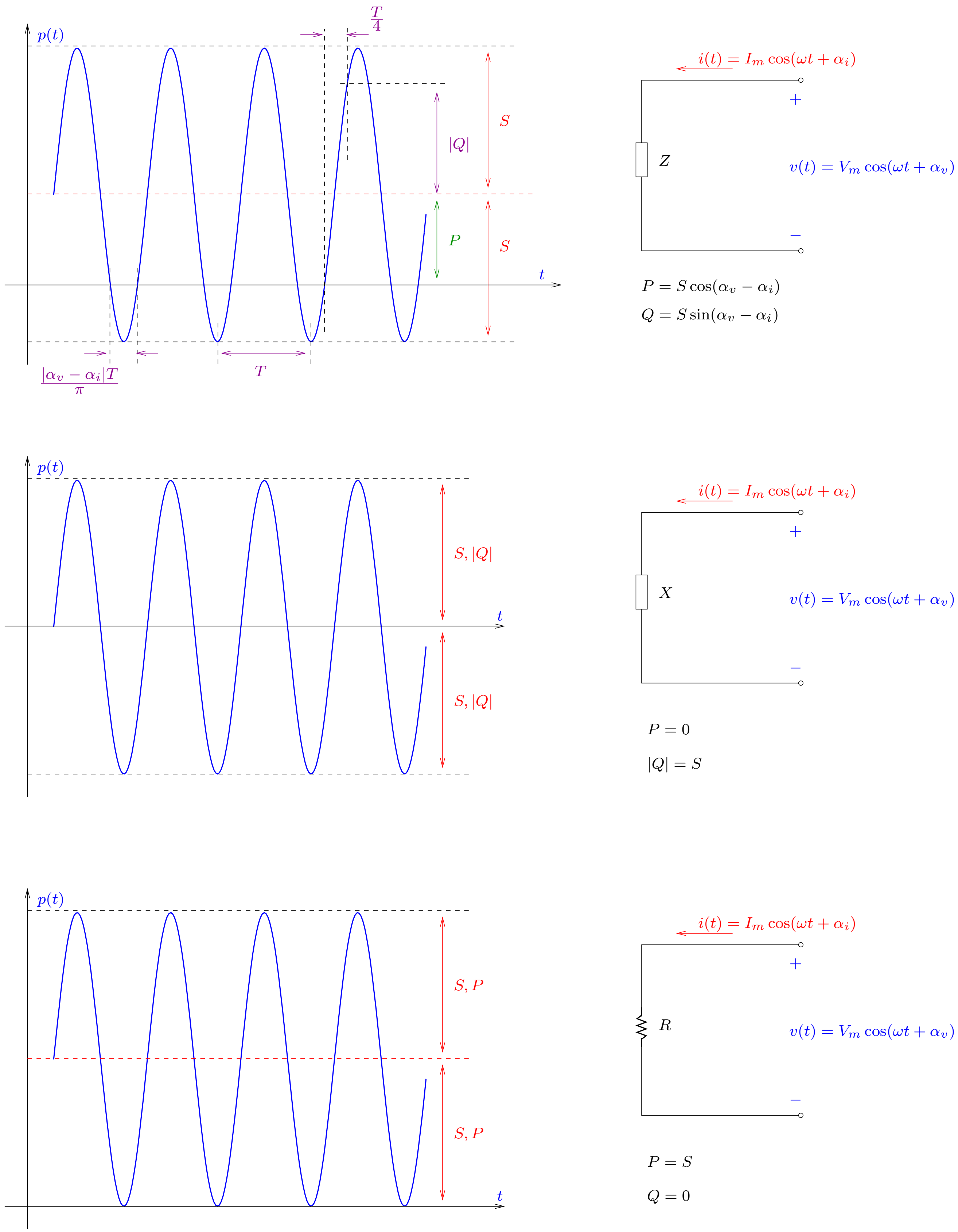
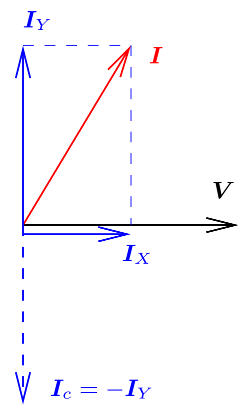



Figure 6: The relationship between the various types of power.

In power factor correction, an additional reactive component is connected in parallel to the load so as to ensure that the overall power factor is 1. Three possible methods that could be used to calculate the value of the additional component are as follows.




- Method 1: Find the load current $\boldsymbol{I}$ in the frequency domain. Let $\boldsymbol{I}_Y$ be the component of $\boldsymbol{I}$ perpendicular on the load voltage $\boldsymbol{V}$. Find a reactive component that has the current $-\boldsymbol{I}_Y$ when the load voltage $\boldsymbol{V}$ is applied to it.
- Method 2: Find the admittance of the load in the form $\boldsymbol{Y} = G + jB$. Find a reactive component of admittance $-jB$.
- Method 3: Find the reactive power of the load $Q$. Find a reactive component that absorbs the reactive power $-Q$ when the load voltage is applied to it.

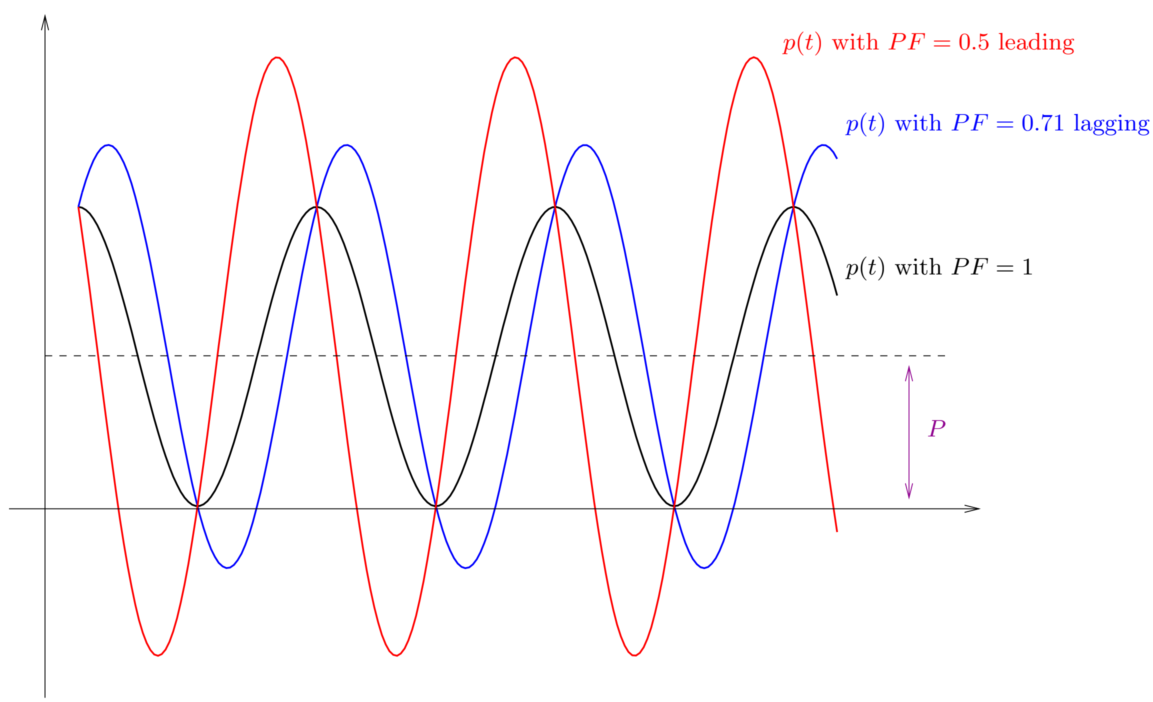



Figure 7: $p(t)$ curves with the same average power $P$ but different power factors. The smallest curve has $PF = 1$ and is optimal.

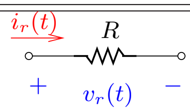
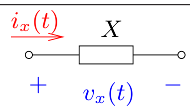
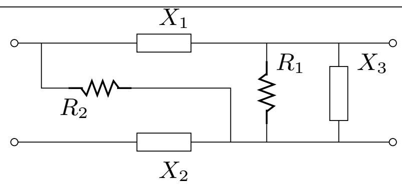
**The Complex Power.** A convenient way to represent the apparent, average, and reactive powers is by means of the complex power. Consider an $n$-terminal component or network. Assuming the notation of Figure 1(b), let $\boldsymbol{V}_{k,rms}$ and $\boldsymbol{I}_{k,rms}$ denote the rms phasor representation of $v_k(t)$ and $i_k(t)$ for $k = 1, 2, \ldots, n$. The complex power is

$$\boldsymbol{S} = \sum_{k=1}^{n} \boldsymbol{V}_{k,rms} \boldsymbol{I}^*_{k,rms}. \tag{37}$$

For the two-terminal component of Figure 1(a), the equation simplifies to

$$\boldsymbol{S} = \boldsymbol{V}_{rms} \boldsymbol{I}^*_{rms}, \tag{38}$$

Table 1: Examples of power calculations.

| | | |
|---|---|---|
|  | $P_R = I^2_{r,rms} R = \frac{V^2_{r,rms}}{R}$ | $Q_R = 0$ |
|  | $P_X = 0$ | $Q_X = I^2_{x,rms} X = \frac{V^2_{x,rms}}{X}$ |
|  | $P = P_{R_1} + P_{R_2}$ | $Q = Q_{X_1} + Q_{X_2} + Q_{X_3}$ |

where $\boldsymbol{V}_{rms}$ and $\boldsymbol{I}_{rms}$ denote the rms phasor representation of $v(t)$ and $i(t)$. Equation (37) can also be written in terms of the real and imaginary parts:

$$\boldsymbol{S} = \sum_{k=1}^{n} V_{k,rms} I_{k,rms} \cos(\alpha_{v,k} - \alpha_{i,k}) + j \sum_{k=1}^{n} V_{k,rms} I_{k,rms} \sin(\alpha_{v,k} - \alpha_{i,k}). \tag{39}$$

Let $\overline{f(t)}$ denote the average of a function $f(t)$. Using a derivation similar to that of equations (7) and (25), the complex power can be written as

$$\boldsymbol{S} = \sum_{k=1}^{n} \overline{v_k(t) i_k(t)} + j \sum_{k=1}^{n} \overline{v_k(t) i_k(t + t_0)}. \tag{40}$$

In view of equations (2) and (22), since $P = \overline{p(t)}$ and $Q = \overline{q(t)}$, we conclude that

$$\boldsymbol{S} = P + jQ \tag{41}$$

Consider a network consisting of $m$ components. Let $\boldsymbol{S}_c$ be the complex power of component $c$, for $c = 1, 2, \ldots, m$. In view of equations (34) and (36), the complex power of the network will be

$$\boldsymbol{S} = \sum_{c=1}^{m} \boldsymbol{S}_c. \tag{42}$$