On a Class of Controlled Invariant Sets

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Outline

This talk presents a new class of controlled invariant sets that can be computed in a non-recursive fashion.

The talk is organized as follows:

1. Controlled and Controllable Invariants
2. A Motivation of Controllable Invariants
3. Computation
4. Results

We will refer to controlled hybrid systems of modes $q \in Q$ and mode dynamics

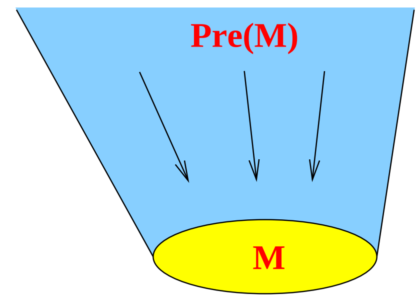
$$x^* = f(q, x, u, d) \tag{1}$$

- x^* is dx/dt for continuous-time and $x(t+1)$ for discrete-time.
- the state is confined to $x \in Inv(q)$.
- $u \in \mathcal{U}(q)$ is the control input.
- $d \in \mathcal{D}(q)$ is the disturbance input.

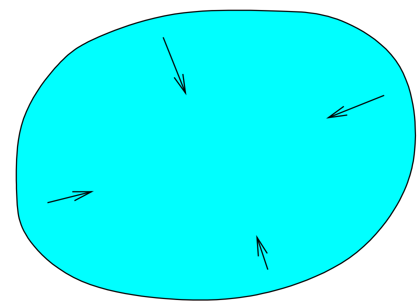
We will focus on the mode dynamics (rather than the hybrid system dynamics).

The computational results are obtained for affine discrete-time dynamics.

Predecessor: Given a set $M \subseteq Inv(q)$, $Pre(M)$ is the set of states $x \in Inv(q)$ from which M can be reached (within the mode, through some feedback control), regardless of the (bounded) disturbances.

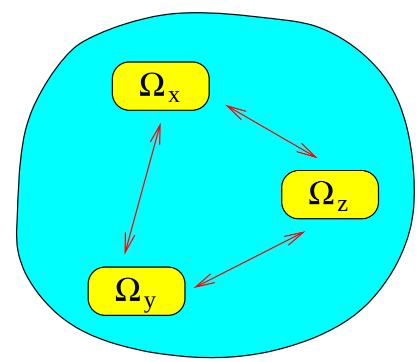


Controlled invariant set: $J \subseteq Inv(q)$ is a controlled invariant set if for all initial states $x_0 \in J$ there is some feedback control keeping the state x in J at all subsequent times, regardless of the (bounded) disturbances.



Given $\Omega \subset \mathbb{R}^n$, let $\Omega_x = \{z \in \mathbb{R}^n | \exists y \in \Omega : z = y + x\}$. Then $I \subseteq Inv(q)$ is a **controllable invariant set** if a connected compact set $\Omega \subset \mathbb{R}^n$ exists such that $0 \in Interior(\Omega)$ and

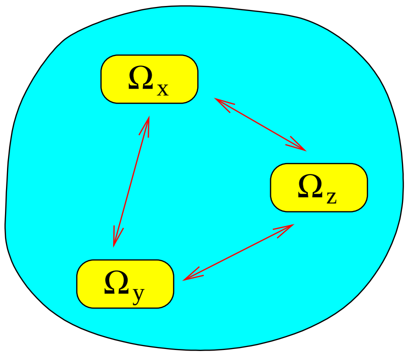
1. $\forall x \in I$: Ω_x is a controlled invariant set.
2. $\forall x_1, x_2 \in I$, $\forall x \in \Omega_{x_1}$: $x \in Pre(\Omega_{x_2})$.
3. $\bigcup_{x \in I} \Omega_x \subseteq Inv(q)$



J is a controlled inv. if for all initial states $x_0 \in J$ there is some feedback control keeping the state x in J at all subsequent times.

I is a controllable inv. if a connected compact set $\Omega \subset \mathbb{R}^n$ exists such that $0 \in Int(\Omega)$ and

1. $\forall x \in I$: Ω_x is a controlled invariant set.
2. $\forall x_1, x_2 \in I$, $\forall x \in \Omega_{x_1}$: $x \in Pre(\Omega_{x_2})$.
3. $\bigcup_{x \in I} \Omega_x \subseteq Inv(q)$



Remarks:

- $\bigcup_{x \in I} \Omega_x$ is a controlled inv. set.
- If J is connected, compact, controlled inv., and $0 \in Int(J)$,$\exists \Omega$ and $\exists$ controllable inv. I such that $J = \bigcup_{x \in I} \Omega_x$: take $I = \{0\}$ and $\Omega = J$.
- However, we are more interested in the cases when Ω is "small" (a neighborhood type Ω).
- For "small" Ω, a controllable inv. identifies a subset of the state space that can always be reached (wherefore the name).
- The range of u is not assumed unbounded!

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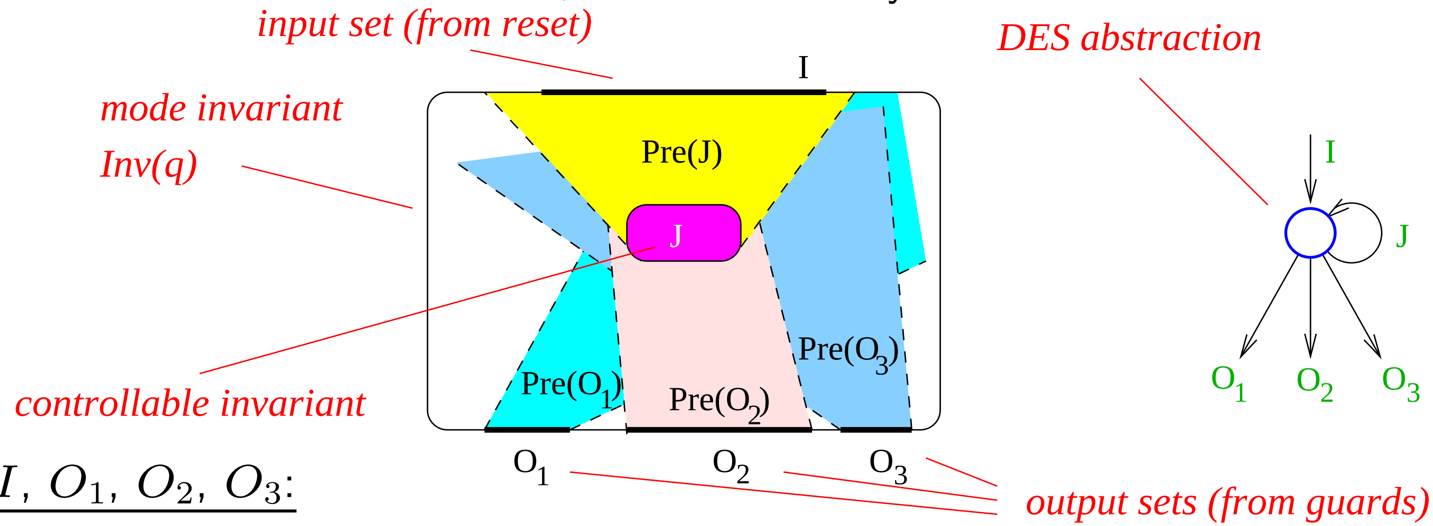
In the abstractions of DES from HS, useful to identify the ideal situation below.



<u>Given I, O_1, O_2, O_3:</u>

Controlled inv. approach:

- Find $Pre(O_1)$, $Pre(O_2)$, $Pre(O_3)$.
- Find J, the max controlled inv. subset of $Pre(O_1) \cap Pre(O_2) \cap Pre(O_3)$
- Check $I \subseteq Pre(J)$ (note that $Pre(J) = J$).

Controllable inv. approach:

- Find J, a max controllable inv. subset of $Inv(q)$.
- Check $\exists x$: $\Omega_x \subset Pre(O_i) \cap J$, $i = 1, 2, 3$ (so full computation of $Pre(O_i)$ may not be needed.)
- Check $I \subseteq Pre(J)$.

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1. We consider hybrid systems with mode dynamics described by

$$x(t+1) = A(q)x(t) + B(q)u(t) + E(q)d(t) \tag{2}$$

Our focus on mode dynamics (rather than overall hybrid system dynamics). The notation is simplified to:

$$x(t+1) = Ax(t) + Bu(t) + Ed(t) \tag{3}$$

2. The control input restricted to $u \in \mathcal{U}$, where $\mathcal{U}$ *is convex.*
3. The disturbance input restricted to $d \in \mathcal{D}$, where $\mathcal{D}$ is such that $\min_{d \in \mathcal{D}} Ed$ and $\max_{d \in \mathcal{D}} Ed$ can be computed (and are finite).
4. The sets Ω are taken of the form $\Omega = \{x : |x| \leq b\}$.

Take Ω of the form $\Omega = \{x : |x| \leq b\}$ (where b is unknown).

The set of points x^* and x_2^* satisfying

$$\exists u(t) \in \mathcal{U} \forall x(t) \in \Omega_{x^*} : x(t+1) \in \Omega_{x_2^*} \tag{4}$$

can be expressed as

$$\exists u \in \mathcal{U}, \forall x \in \Omega_{x^*} : \left\{ \begin{array}{lcl} Ax + Bu + d^+ & \leq & x_2^* + b \\ Ax + Bu - d^- & \geq & x_2^* - b \end{array} \right. \tag{5}$$

where $d^+ = \max_{d \in \mathcal{D}} Ed$ and $d^- = -\min_{d \in \mathcal{D}} Ed$.

Quantifier elimination will be used.

After projecting out u, an expression of the form below is obtained:

$$\forall x \in \Omega_{x^*} : Gx + Hb + Mx_2^* \leq g \quad (6)$$

The remaining quantifier can be also eliminated:

$$(G + M)x^* + (|G| + H)b + M(x_2^* - x^*) \leq g \quad (7)$$

If we require (7) be satisfied for all x_2^* with $|x_2^* - x^*| < \delta$ we obtain:

$$(G + M)x_1^* + (|G| + H)b + |M|\delta \leq g \quad (8)$$

For $\delta = 0$ we get:

$$(G + M)x^* + (|G| + H)b \leq g \quad (9)$$

We show that

$$(G + M)x^* + (|G| + H)b < g$$

is a nearly optimal controllable invariant set.

Assuming $Inv(q) = \mathbb{R}^n$**:**

Proposition 1. $J = \{x : (G + M)x + (|G| + H)b < g\}$ is a controllable invariant of set $\Omega = \{x : |x| \le b\}$.

Proposition 2. All controllable invariants of set $\Omega = \{x : |x| \le b\}$ are subsets of $\overline{J}$, the closure of J.

Assuming $Inv(q) \subset \mathbb{R}^n$**:**

Let $W = \{x \in Inv(q) : \Omega_x \subseteq Inv(q)\}$. *Assume W is connected.* Then:

Proposition 3. $J \cap W$ is a controllable invariant of set $\Omega = \{x : |x| \le b\}$.

Proposition 4. All controllable invariants of set $\Omega = \{x : |x| \le b\}$ are subsets of $\overline{J} \cap W$.

Proposition 5. A nonempty controllable invariant of set $\Omega = \{x : |x| \leq b\}$ exists if and only if $\overline{J} \cap W \neq \emptyset$.

The following results no longer restrict controllable invariants to $\Omega = \{x : |x| \leq b\}$:

Proposition 6. The controllability of (A, B) is neither sufficient nor necessary for the existence of a nonempty controllable invariant set.

Proposition 7. Assume $0 \in \mathcal{D}$. Then, a nonempty controllable invariant exists only if all uncontrollable eigenvalues λ of (A, B) satisfy $|\lambda| \leq 1$.

Example

Assume

$$x(t+1) = ax(t) + u(t) + d(t) \tag{10}$$

$a \in \mathbb{R}$, $d^+ = d^- = d_0$, $-u_0 \leq u \leq u_0$ and $Inv(q) = \mathbb{R}$. We start with (5):

$$\exists u \in \mathcal{U}, \forall x \in \Omega_{x_1^*} : \begin{cases} ax + u + d_0 & \leq & x_2^* + b \\ -ax - u + d_0 & \leq & -x_2^* + b \end{cases} \tag{11}$$

and eventually obtain the controllable invariant set

$$\begin{cases} (a-1)x + (|a|-1)b + d_0 & < & u_0 \\ (-a+1)x + (|a|-1)b + d_0 & < & u_0 \end{cases} \tag{12}$$

for a b such that $b \geq d_0$ and $(|a|-1)b + d_0 < u_0$.

There is no solution unless $|a|d_0 < u_0$ or $|a| < 1$.

Final Remarks

1. The approach is not recursive!
2. The only complex operation is the elimination of the control input (the worst case complexity of Fourier-Motzkin elimination is double-exponential).
3. The result is nearly optimal, as all controllable invariants with the given set Ω are subsets of the closure of the computed set.
4. How to select b? It can be shown it must satisfy $2b \geq d^+ + d^-$.
5. Controlled invariant sets can be obtained easily from controllable invariant sets
   (a) recall, if I is a controllable inv., $J = \bigcup_{x \in I} \Omega_x$ is controlled inv.
   (b) the union of J's for all admissible b's is a controlled invariant. It can be computed by eliminating b.