

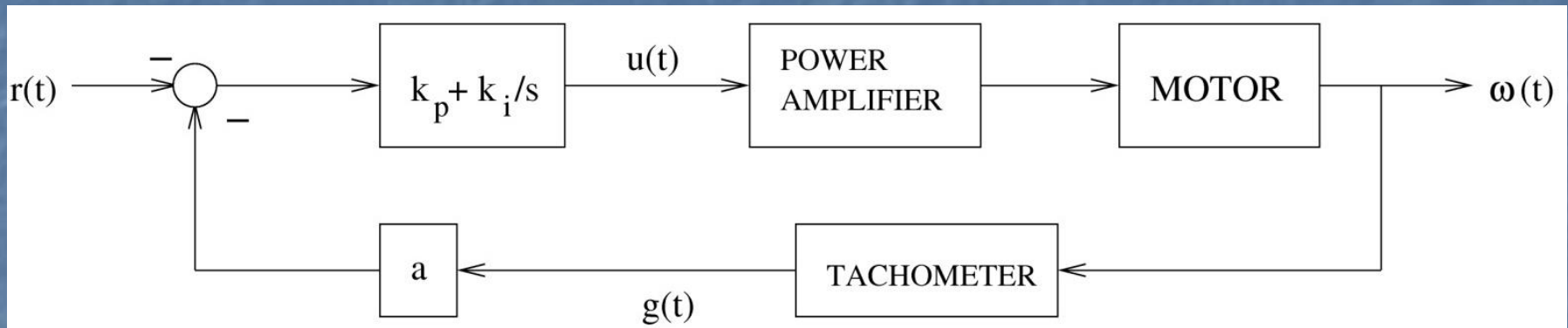
Approximating Transfer Functions

How to Obtain a Lower Order Approximate Transfer Function

Higher Order Systems

- Physical systems have poles.
 - Because of poles, the output will be negligible when the frequency of the input is high enough.
- The order of a system equals the number of poles.
- Consider time response specifications.
 - The previous lectures show how to design controllers for first and second order systems.
 - How about systems of order $n > 2$?
 - Commonly, systems have $n > 2$!

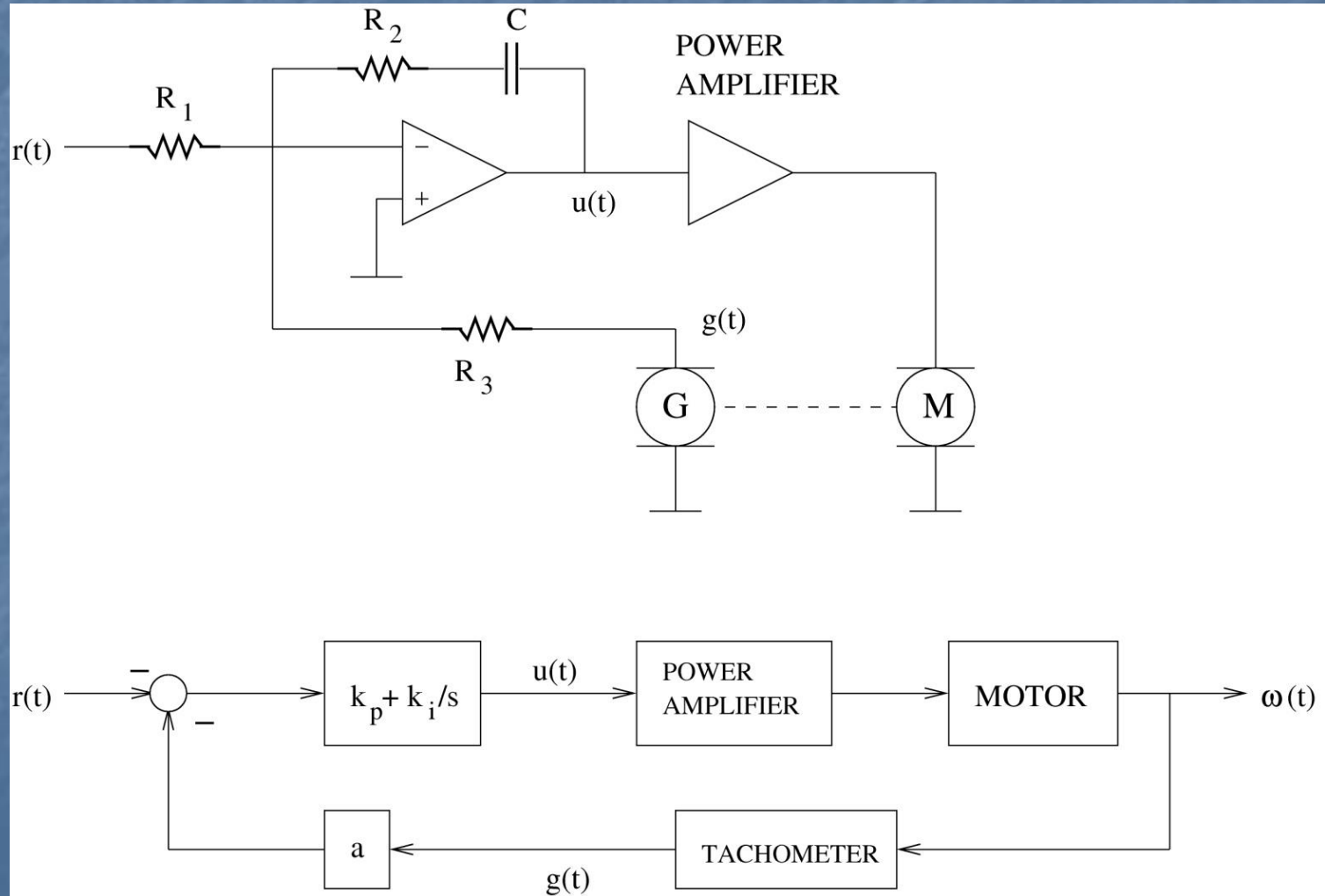
Higher Order Systems—Example



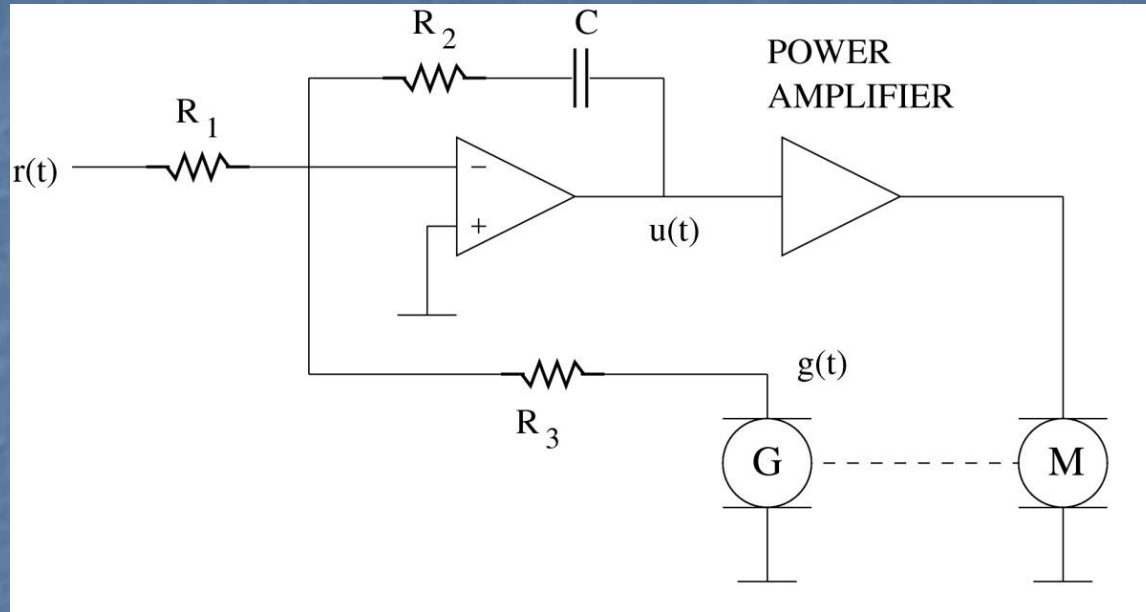
- Consider a simple speed control system.
- The PI controller has one pole.
- The power amplifier has at least one pole.
- The motor has one pole.
- The total number of poles is $n \geq 3$.

Higher Order Systems—Example

- Consider an analog implementation of the controller.

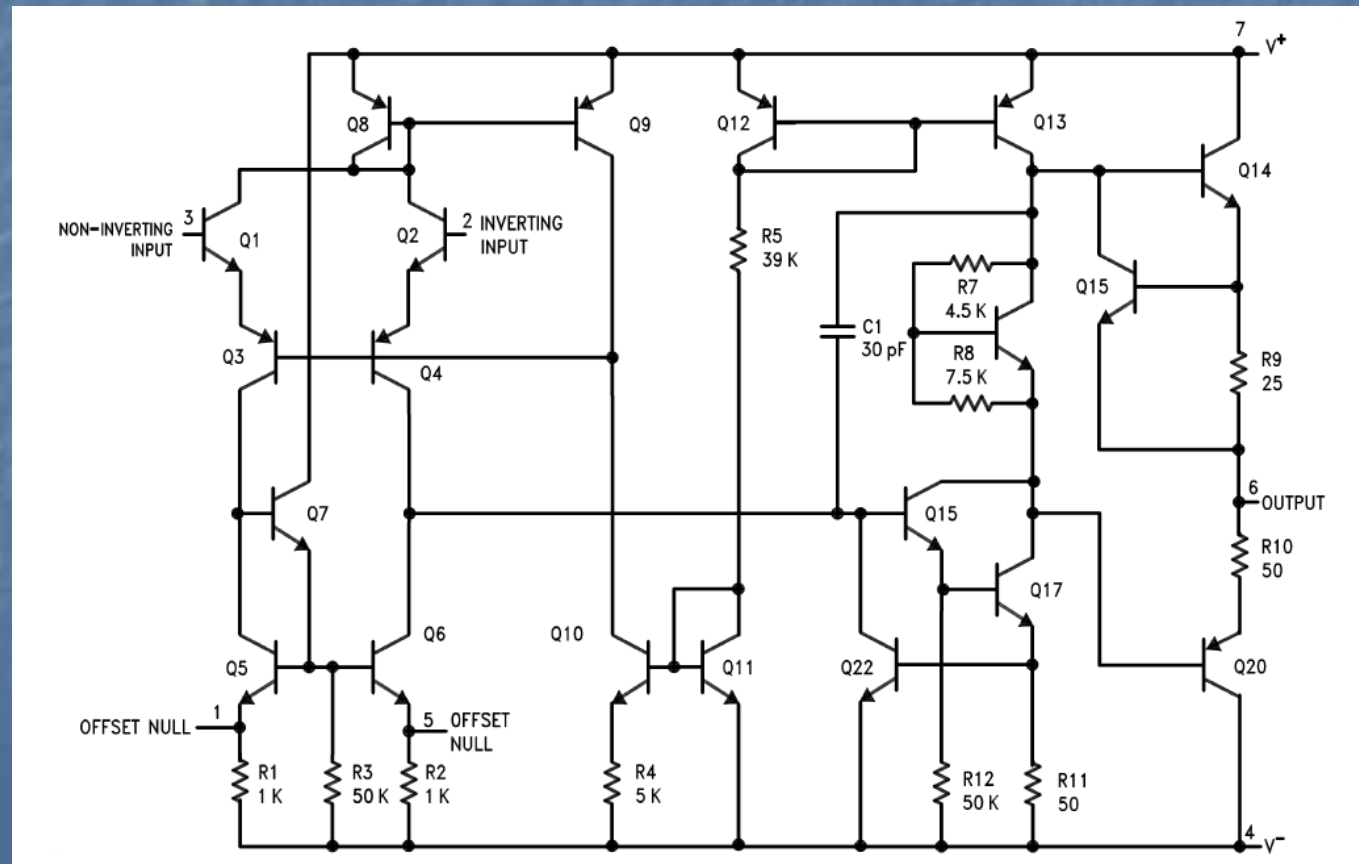


Higher Order Systems—Example



- The PI controller and the summing junction are implemented by the operational amplifier.
- The operational amplifier adds poles.
- Clearly, the total number of poles is $n \geq 4$.

- The figure shows the implementation of a classic operational amplifier (741).
- There are 22 transistors.
- Transistors have poles ...
- Note that operational amplifiers have multiple poles!

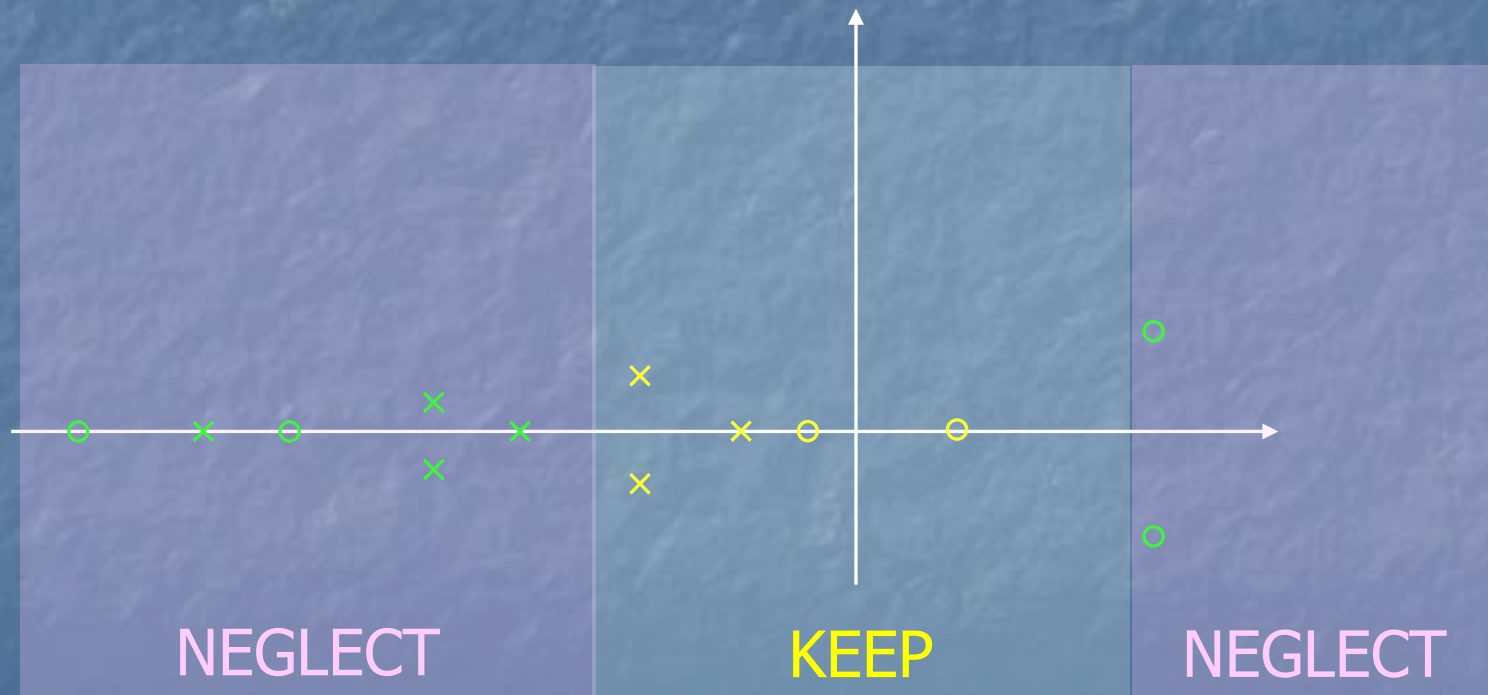


Higher Order Systems

- How to design controllers for time response specifications when $n > 2$?
- It turns out that the time response is determined by the **dominant pole(s)**.
 - The dominant pole is the pole (or pair of complex poles) closest to the imaginary axis.
- Therefore, approximate the system as a first order system or as a second order system.
 - The approximation will include the dominant pole(s) and neglect the poles and zeros that are considerably further from the imaginary axis.

Approximations—Principle

- Assume a stable system.
- The poles and zeros sufficiently close to the imaginary axis are kept; the others are neglected.



Approximations—Principle

- Assume that the highlighted terms of $H(s)$ will be neglected.

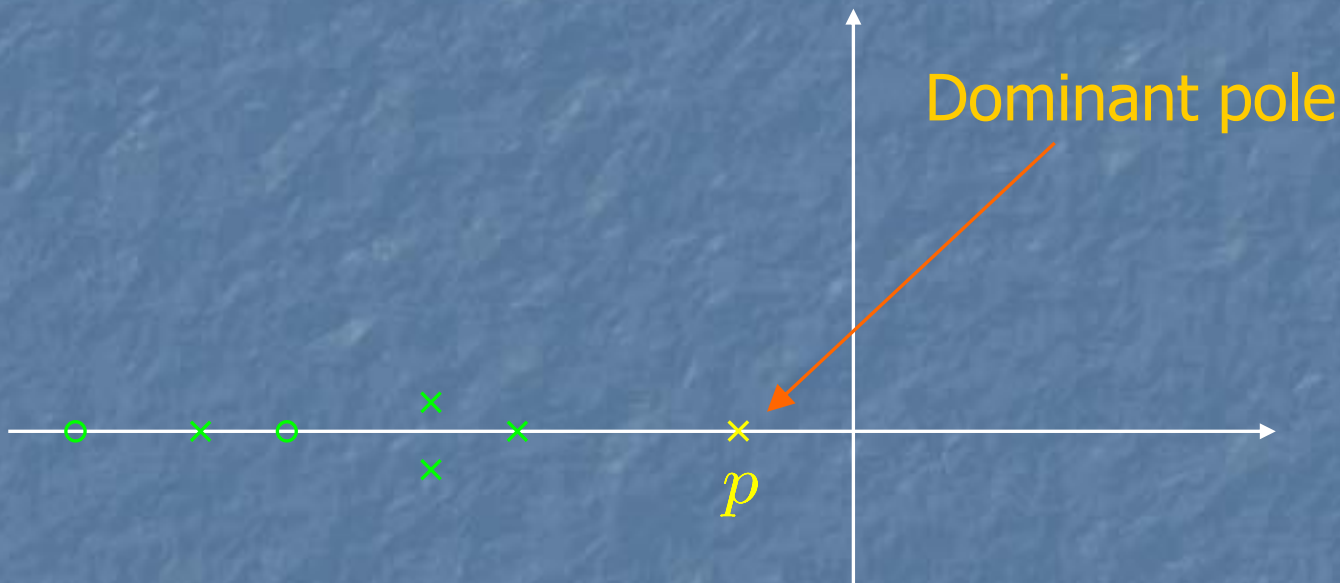
$$H(s) = \frac{c(s - z_1) \dots (s - z_u)(s - z_{u+1}) \dots (s - z_m)}{(s - p_1) \dots (s - p_v)(s - p_{v+1}) \dots (s - p_n)}$$

- The approximate model $H_a(s)$ is obtained by replacing the neglected terms with their DC gain:

$$H_a(s) = \frac{c(s - z_1) \dots (s - z_u)(-z_{u+1}) \dots (-z_m)}{(s - p_1) \dots (s - p_v)(-p_{v+1}) \dots (-p_n)}$$

Dominant Poles

- Consider the pole-zero plot of **stable** system.



- If the system has a single dominant pole, a first order approximation could be appropriate:

$$H(s) \simeq \frac{A}{s - p}$$

Dominant Poles

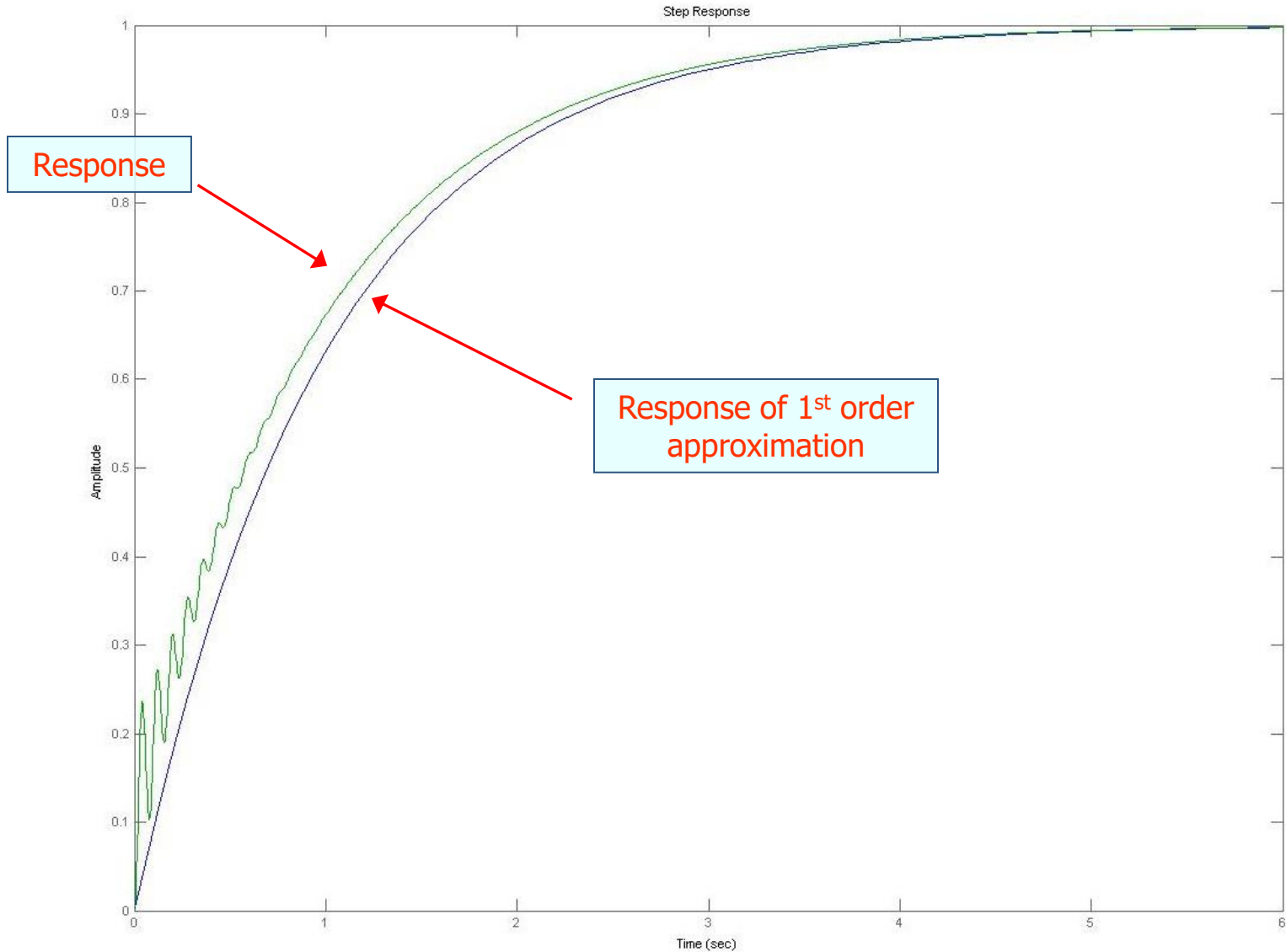
- With a single dominant pole p :

$$H(s) \simeq H_a(s) = \frac{A}{s - p}$$

- Often, the values of the neglected poles and zeros are unknown.
- Nonetheless, we can still find A by imposing the condition that $H(s)$ and $H_a(s)$ have the same steady state response to a step input.
- Then, from the condition $H_a(0) = H(0)$ we derive:

$$H(s) \simeq \frac{H(0)|p|}{s - p}$$

First order approximation



Approximation—Example

Find a first order approximation based on the green curve shown on the previous slide. Assume the curve is the unit-step response.

- Since $y_{ss} = H(0)r_{ss}$ and $y_{ss} = r_{ss} = 1$, we derive $H(0) = 1$.
- The first order approximation has the form

$$H(s) \simeq \frac{H(0)|p|}{s - p} = \frac{|p|}{s - p}$$

- From the first order system theory, the settling time is

$$t_s \simeq \frac{4}{-p}$$

- The graph indicates $t_s \simeq 4$. Therefore, $p = -1$ and

$$H(s) \simeq \frac{1}{s + 1}$$

Approximation—Example

Find the 2% settling time of the step response of

$$H(s) = \frac{s + 20}{(s + 2)(s + 10)(s + 25)}$$

- The system has poles at -2 , -10 , and -25 .
- The system has also a zero at -20 .
- The dominant pole is $p = -2$.
- A first order approximation is

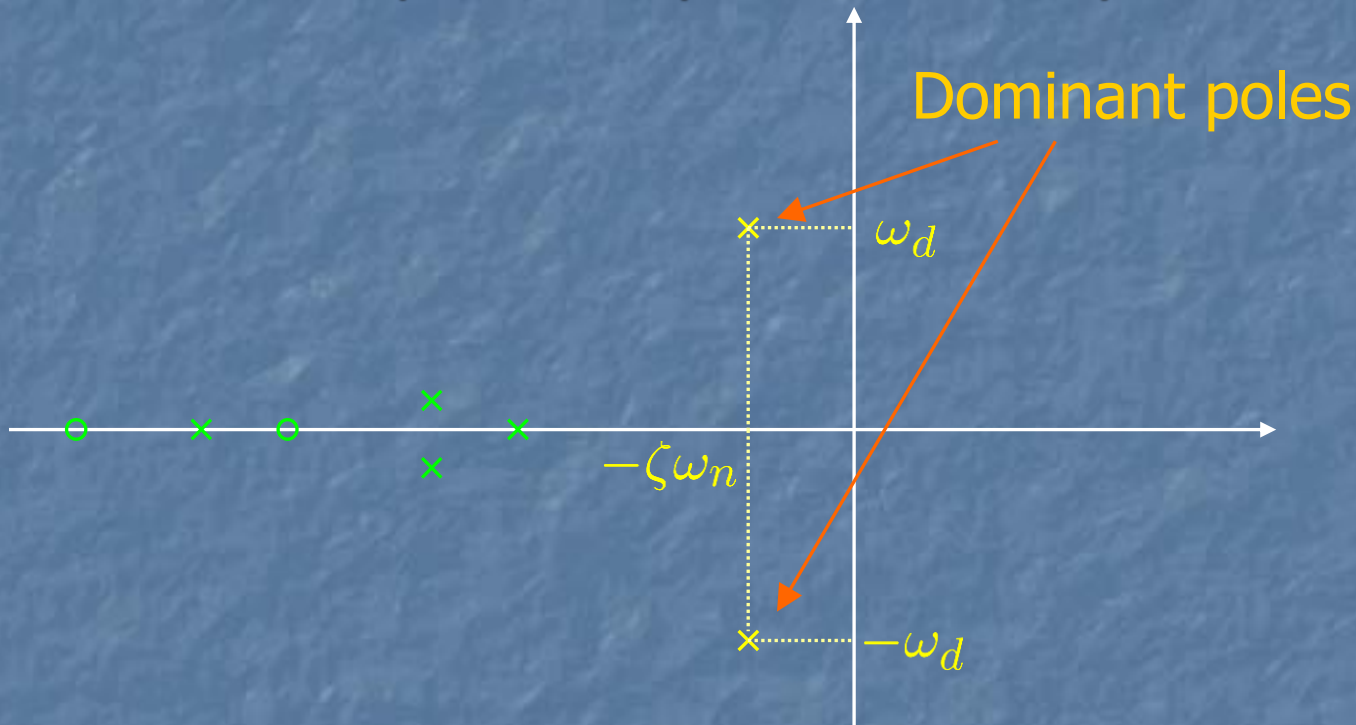
$$H(s) \approx \frac{0 + 20}{(s + 2)(0 + 10)(0 + 25)} = \frac{0.08}{s + 2}$$

- From the first order system theory, the settling time is

$$t_s \approx \frac{4}{-p} = 2 \text{ sec}$$

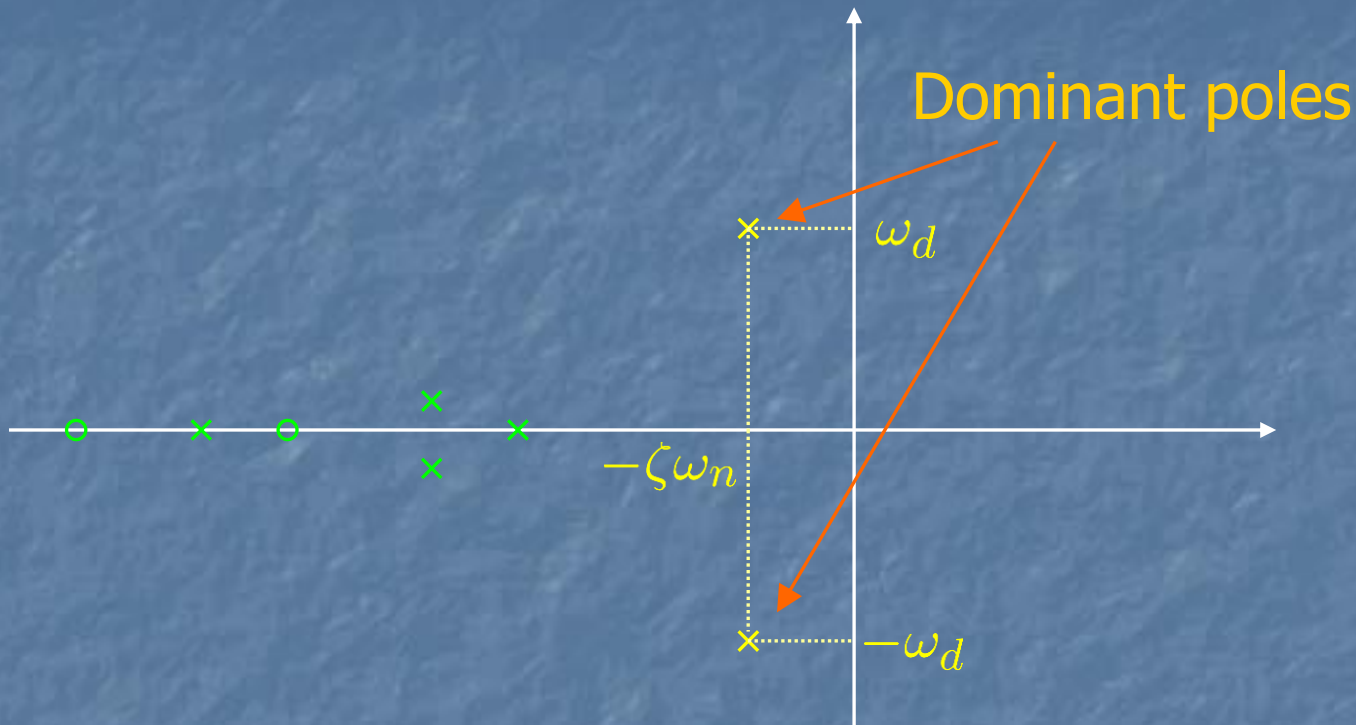
Dominant Poles

- Consider the pole-zero plot of **stable** system.



- If the system as a pair of complex dominant poles, a second order approximation could be appropriate.

Dominant Poles



- Let p be a dominant pole.
- The step response will have, approximately, the damping factor ζ and the natural frequency ω_n , where

$$\omega_n = |p| \text{ and } \zeta\omega_n = -\text{Real}(p)$$

Dominant Poles

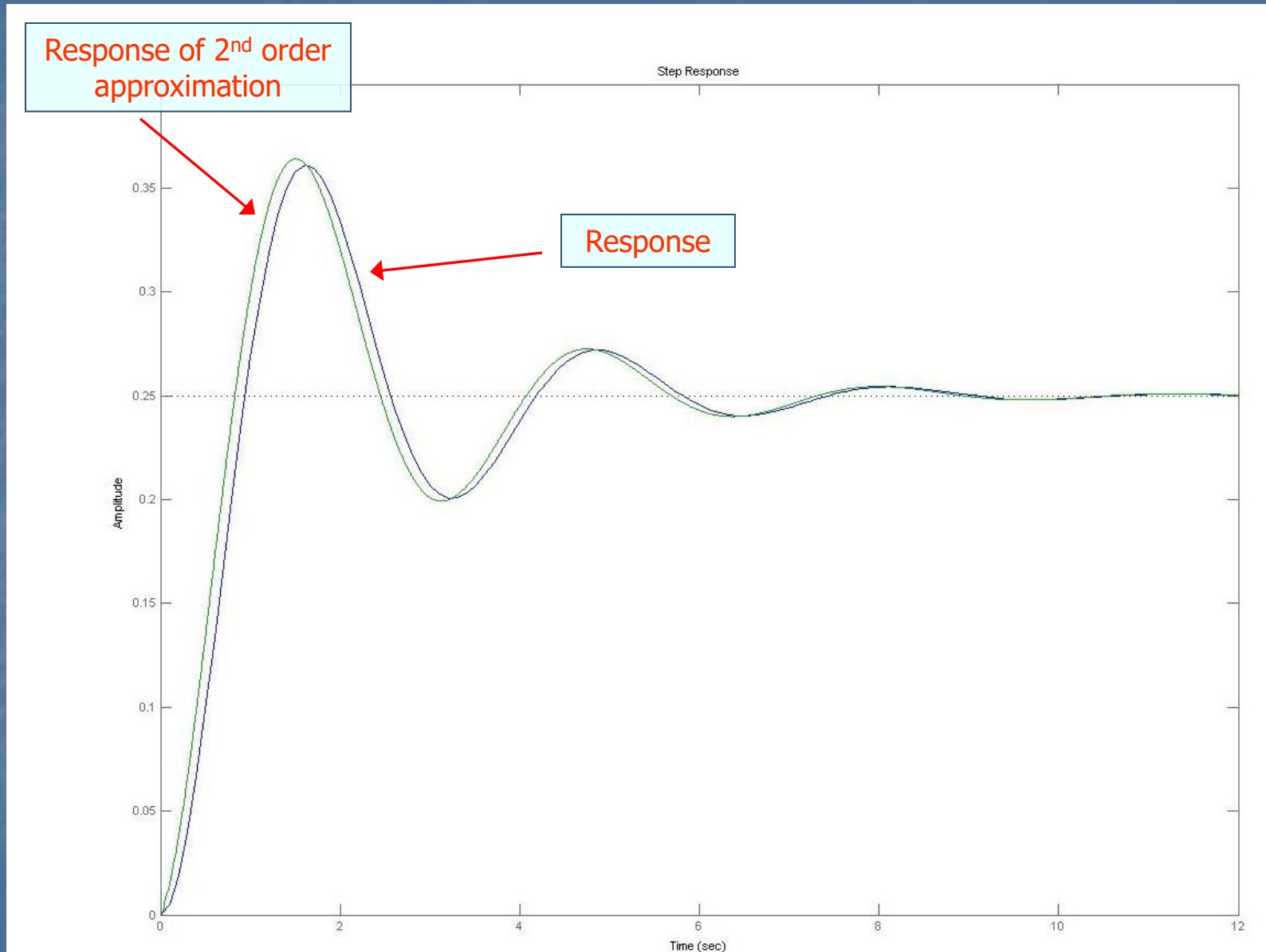
- In a second order approximation:

$$H(s) \simeq H_a(s) = \frac{A}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Since $H(s)$ and $H_a(s)$ should have the same steady state response to a step input, the constant A will be found so that $H_a(0) = H(0)$.
- We derive:

$$H(s) \simeq \frac{H(0)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second-order approximation



Approximation—Example

Find the 2% settling time to a step input of

$$H(s) = \frac{s + 30}{(s^2 + 4s + 8)(s + 10)}$$

- The system has poles at $-2 \pm 2j$ and at -10 .
- There is also a zero at -30 .
- The dominant poles are $-2 \pm 2j$.
- The approximation is:

$$H(s) \approx \frac{0 + 30}{(s^2 + 4s + 8)(0 + 10)} = \frac{3}{s^2 + 4s + 8}$$

- From the theory of second order systems:

$$t_s \approx \frac{4}{\zeta \omega_n} = \frac{4}{2} = 2 \text{ sec}$$

Approximation—Example

Find a lower order approximation of

$$H(s) = \frac{s - 2}{(s^2 + 4s + 8)(s + 10)}$$

- The system has poles at $-2 \pm 2j$ and at -10 .
- There is also a zero at 2.
- The dominant poles are $-2 \pm 2j$.
- However, the zero is equally close to the imaginary axis. Clearly, it cannot be neglected!

$$H(s) \simeq \frac{s - 2}{(s^2 + 4s + 8)(0 + 10)} = \frac{0.1(s - 2)}{s^2 + 4s + 8}$$

Justifying the Approximations

- Assume

$$H(s) = \frac{c(s - z_1) \dots (s - z_u)(s - z_{u+1}) \dots (s - z_m)}{(s - p_1) \dots (s - p_v)(s - p_{v+1}) \dots (s - p_n)}$$

- Consider an approximation in which the poles p_{v+1}, \dots, p_n and the zeros z_{u+1}, \dots, z_m are neglected:

$$H_a(s) = \frac{c(s - z_1) \dots (s - z_u)(-z_{u+1}) \dots (-z_m)}{(s - p_1) \dots (s - p_v)(-p_{v+1}) \dots (-p_n)}$$

- Consider also the unit step response:

$$Y(s) = H(s) \cdot \frac{1}{s} \text{ and } Y_a(s) = H_a(s) \cdot \frac{1}{s}$$

- For a reasonable approximation, $y(t) \simeq y_a(t)$.

Justifying the Approximations

$$Y(s) = \frac{B_0}{s} + \frac{B_1}{s - p_1} + \dots + \frac{B_v}{s - p_v} + \dots + \frac{B_n}{s - p_n}$$

$$Y_a(s) = \frac{B'_0}{s} + \frac{B'_1}{s - p_1} + \dots + \frac{B'_v}{s - p_v}$$

- If $|p_1|, |p_2|, \dots, |p_v| \ll |p_{v+1}|, \dots, |p_n|, |z_{u+1}|, \dots, |z_m|$, then

$$p_i - z_j \simeq -z_j \text{ for all } i = 1 \dots v \text{ and } j = u + 1 \dots m$$

$$p_i - p_j \simeq -p_j \text{ for all } i = 1 \dots v \text{ and } j = v + 1 \dots n$$

Note that $B'_0 = B_0, B'_1 \simeq B_1, \dots, B'_v \simeq B_v$.

Justifying the Approximations

$$y(t) = B_0 + B_1 e^{p_1 t} + \dots + B_v e^{p_v t} + \dots + B_n e^{p_n t}$$

$$y_a(t) = B'_0 + B'_1 e^{p_1 t} + \dots + B'_v e^{p_v t}$$

- Since $|p_1|, |p_2|, \dots, |p_v| \ll |p_{v+1}|, \dots, |p_n|$, assuming p_{v+1}, \dots, p_n in the LHP, $B_j e^{p_j t}$ decays much faster than $B_i e^{p_i t}$ for all $i = 1 \dots v$ and $j = v + 1 \dots n$.
- Accounting also for $B'_0 = B_0, B'_1 \simeq B_1, \dots, B'_v \simeq B_v$, it follows that

$$y_a(t) \simeq y(t).$$