The Time Response

First and Second Order Systems.

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Control Systems Design

Involves several steps:

1. Find models for plant, sensors, actuators.
2. Select control structure, based on specifications (feedforward, feedback, …).
3. Select controller type (PID, lead, lag, …).
4. Design controller.
5. If specification cannot be satisfied, go to step 2.
6. Tune controller parameters based on system simulation.
7. Tune controller parameters on the real plant.

Specifications

So far we have studied

- Stability

  any bounded input → bounded output.

- Steady state error

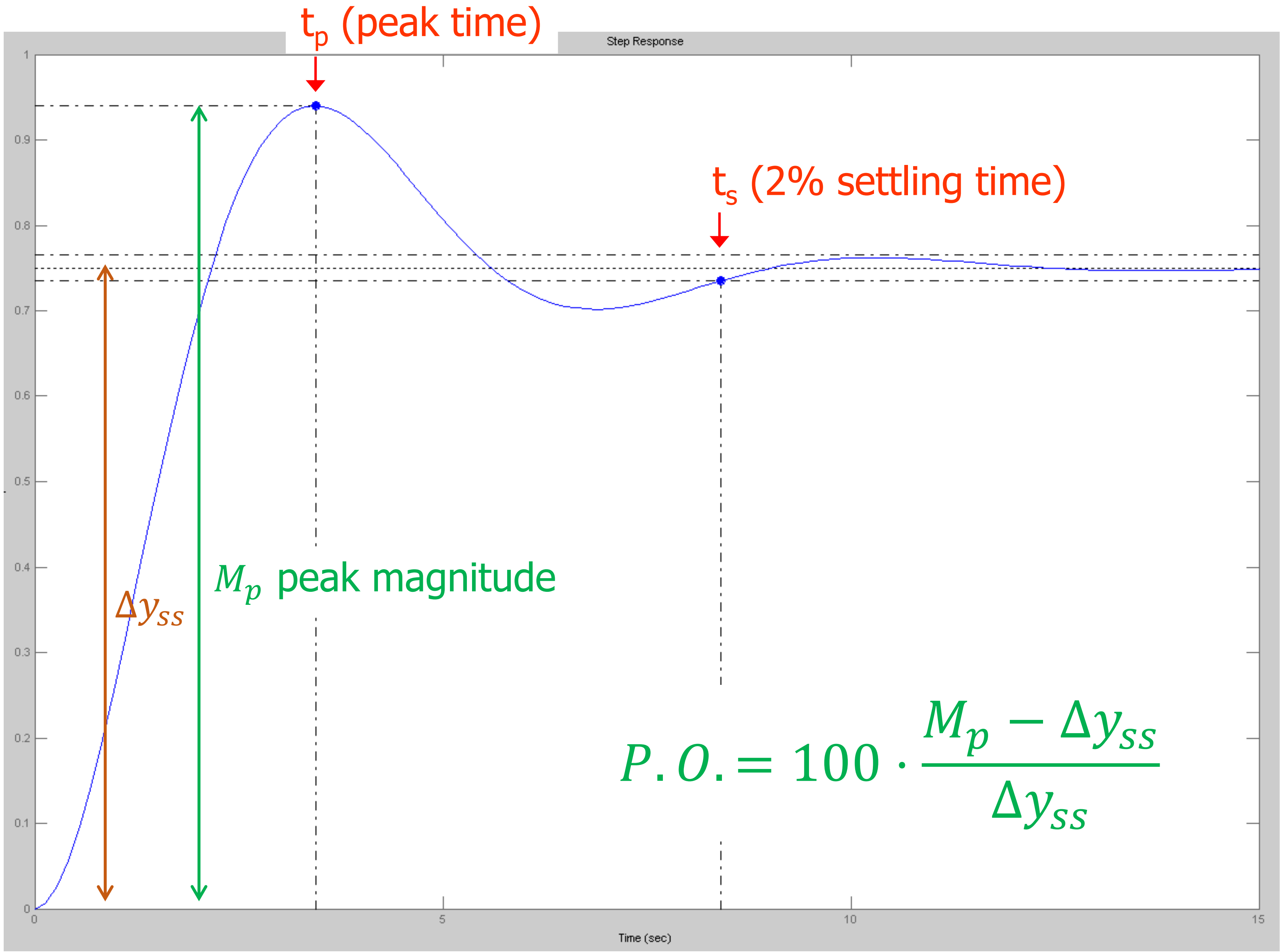
  how much the steady state may deviate from its prescribed value under constant disturbances.

At this time we will consider

- Time Response Specifications

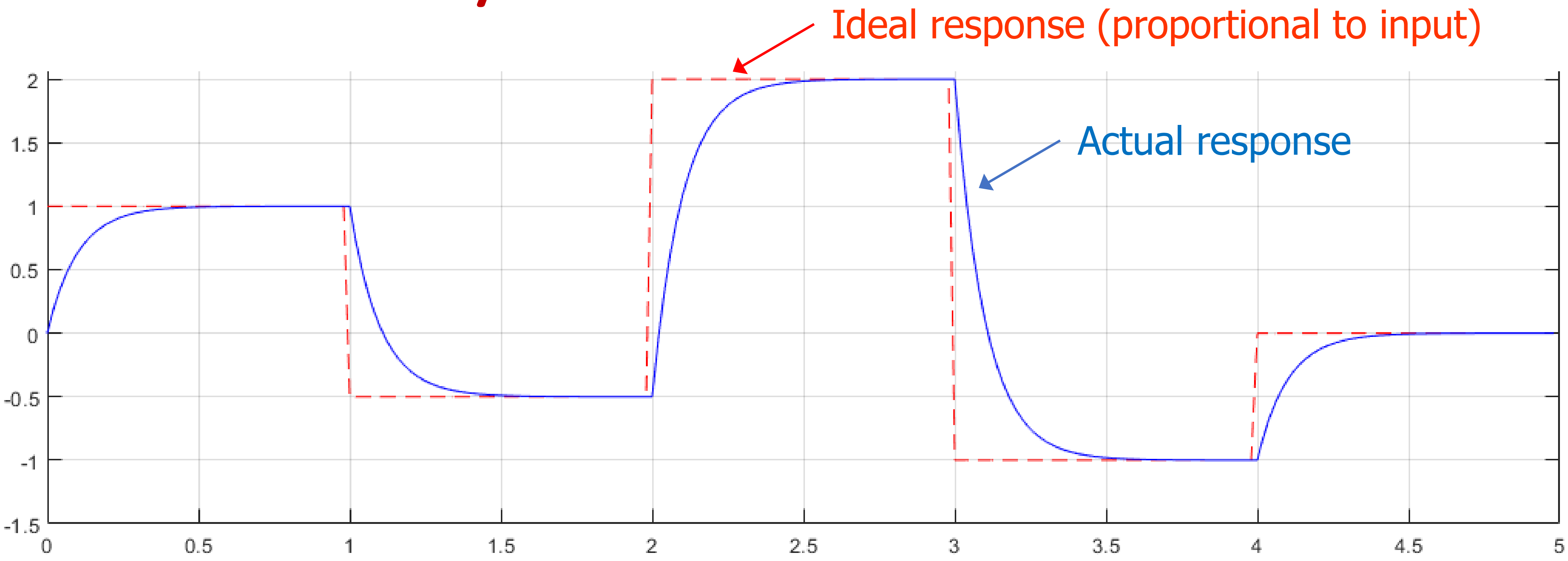
Typical Time Response Parameters

- Settling time t_s
- Peak time t_p
- Percent overshoot (PO).
- Steady-state value y_{ss}



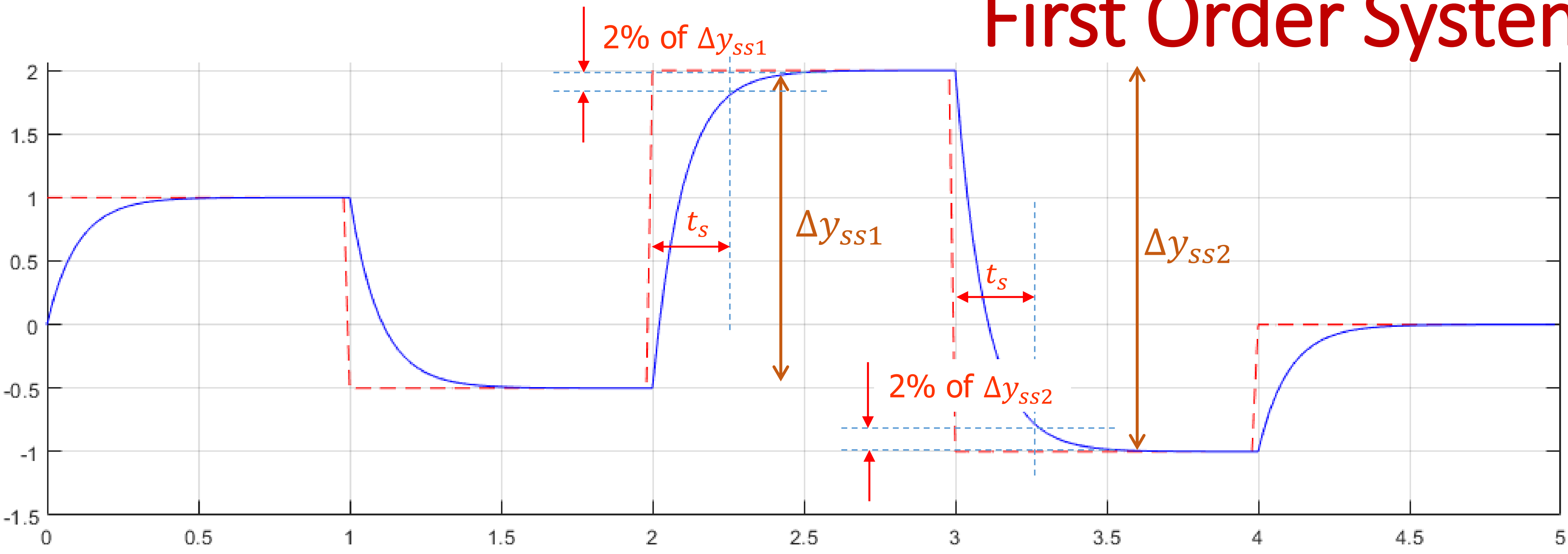
$$P.O. = 100 \cdot \frac{M_p - \Delta y_{ss}}{\Delta y_{ss}}$$

First Order Systems



- Assume that the input of a first order system changes in steps.
- The output will not overshoot ($PO = 0\%$) and will have no peak time.
- The settling time describes the response.

First Order Systems



A first order system is described by a *transfer function* of the form

$$H(s) = \frac{A}{s+p}$$

where A and $p > 0$ are constants of the system. The *2% settling time* is

$$t_s \simeq \frac{4}{p}$$

First Order Systems

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Proof outline: An input that changes in steps is a sum of (delayed) step inputs.

The response of $H(s)$ to a single step input of amplitude m and delay t_0 is:

$$y(t) = \frac{A \cdot m}{p}\left(1 - e^{-p(t-t_0)}\right)$$

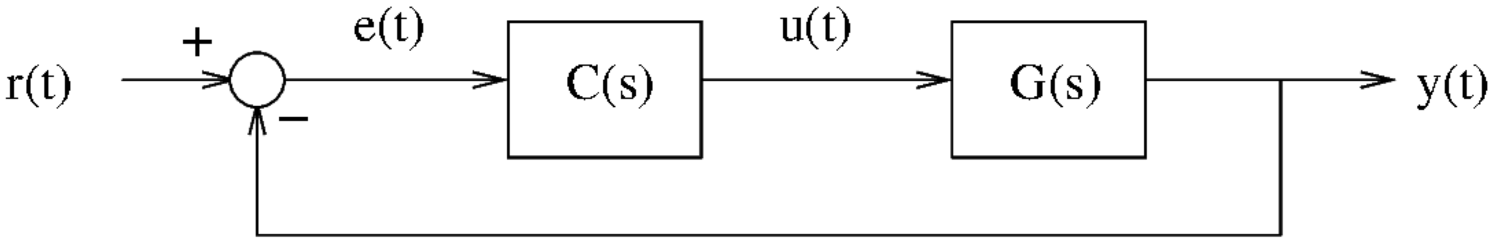
When $t - t_0 = t_s$, the output is at 2% from the steady state.

$$\frac{A}{p}(1 - 0.02) = \frac{A}{p}(1 - e^{-pt_s}) \Rightarrow t_s = -\frac{\ln(0.02)}{p} \simeq \frac{4}{p}$$

First Order Systems—Examples

- Let v be the velocity of a vehicle and τ the torque of the motor. The transfer function $\frac{V(s)}{T(s)}$ corresponds, approximately, to a first order system.
- Common operational amplifiers can be approximated by first-order systems.
- An RL or RC circuit.

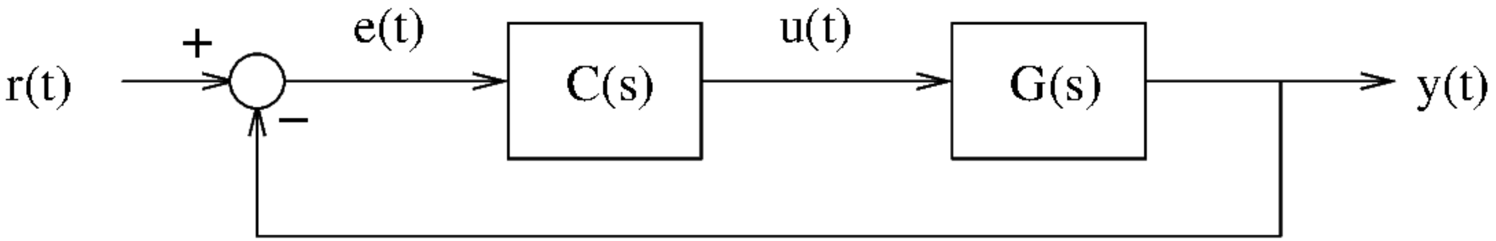
First Order Systems—Example



Assume $G(s) = \frac{0.1}{s+5}$*. Find a controller that ensures that the settling time to a step input is 10 times smaller than the settling time of the plant.*

- The closed-loop transfer function is $CL(s) = \frac{CG}{1+CG} = \frac{0.1C}{s+5+0.1C}$.
- Let $C(s) = k$. For stability, $5 + 0.1k > 0$.
- If $C(s) = k$, the closed-loop is a first-order system. Therefore, $t_s \simeq \frac{4}{5+0.1k}$.
- The settling time of the plant to a step input is $t_{sp} \simeq \frac{4}{5} = 0.8$ sec.
- $t_s = 0.1t_{sp} \Rightarrow k = 450$. The answer is $C(s) = 450$.

First Order Systems



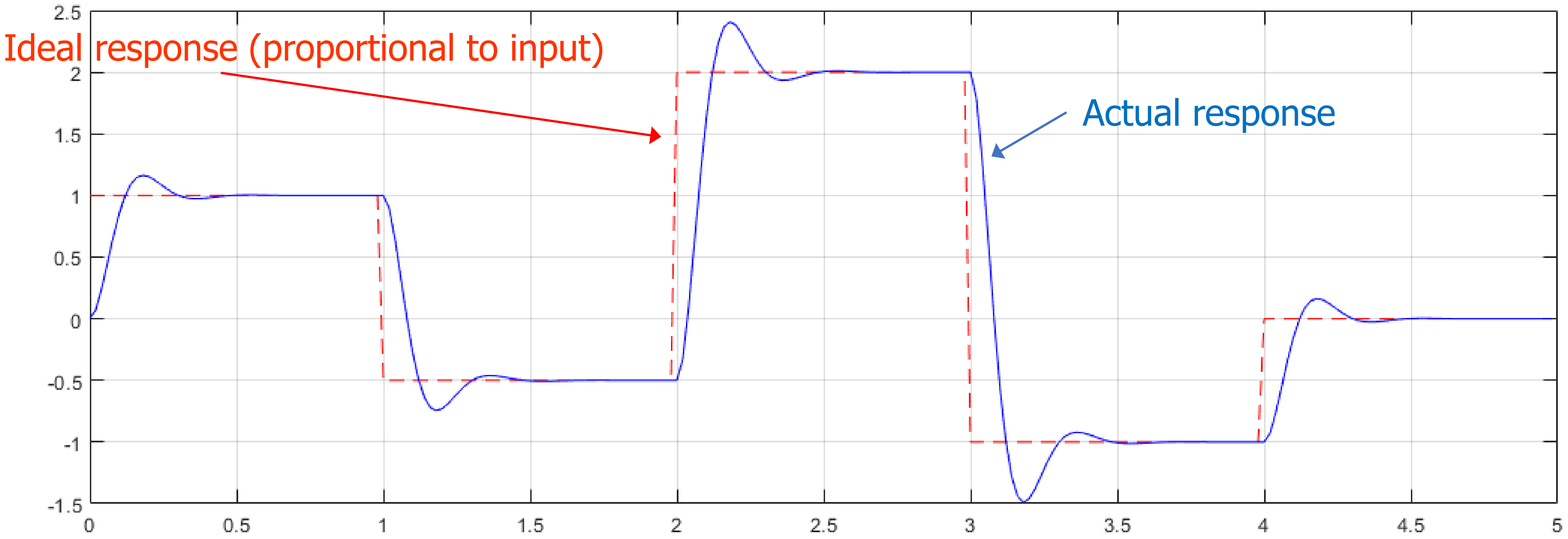
If $C(s) = k$ and $G(s)$ is a first order system, $\lim_{k\to\infty} t_s = 0$.

Consequently, any settling time can be achieved with a high enough value of k!

From a practical viewpoint, large values of k will create several issues:

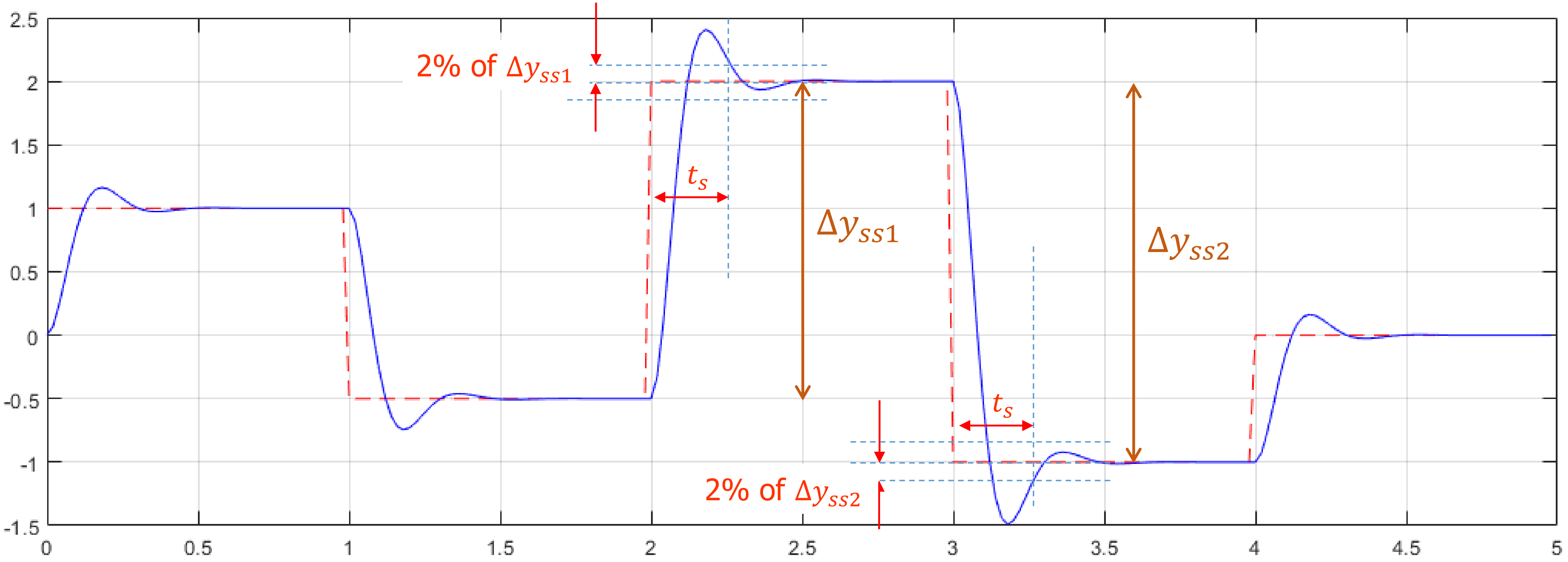
- Instability: Plants are only approximately first-order systems. Because of neglected high frequency poles, a large enough gain will lead to instability.
- Actuator saturation:
  - Consider a unit step input. Assume $y(0) = 0$.
  - At time $t = 0$, when the step is applied, $r(0) = 1$, so $e(0) = 1$ and $u(0) = k$.
  - Practical actuators cannot apply the control $u(0) = k$ unless k is small enough.

Second Order Systems



- They can overshoot when the input changes in steps.
- Two important parameters: the settling time and the percent overshoot.

Second Order Systems



- The 2% settling time is the time until the output stays within $\pm 2\%$ of the step from its previous steady-state value.

Second Order Systems

- Consider a second-order system described by a transfer function

$$H(s) = \frac{A}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  - A is a constant.
  - ζ is the damping ratio. It is dimensionless.
  - $\omega_n > 0$ is the natural frequency. It is measured in rad/s (like the s variable).
- If $\zeta \leq 0$, the system is unstable.
- If $0 < \zeta < 1$, the step response will overshoot.
- If $\zeta \geq 1$, the step response will resemble the response of a first order system; it will have no overshoot.
- If $\zeta = 1$, the system is said to be critically damped.

Second Order Systems

- Assume $0 < \zeta < 1$.
- The unit-step response is

$$y(t) = \frac{A}{\omega_n^2}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \varphi)\right)$$

where

  - $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.
  - $\varphi = \cos^{-1}\zeta$.

Second Order Systems

- Assume $0 < \zeta < 1$.
- The 2% settling time is:

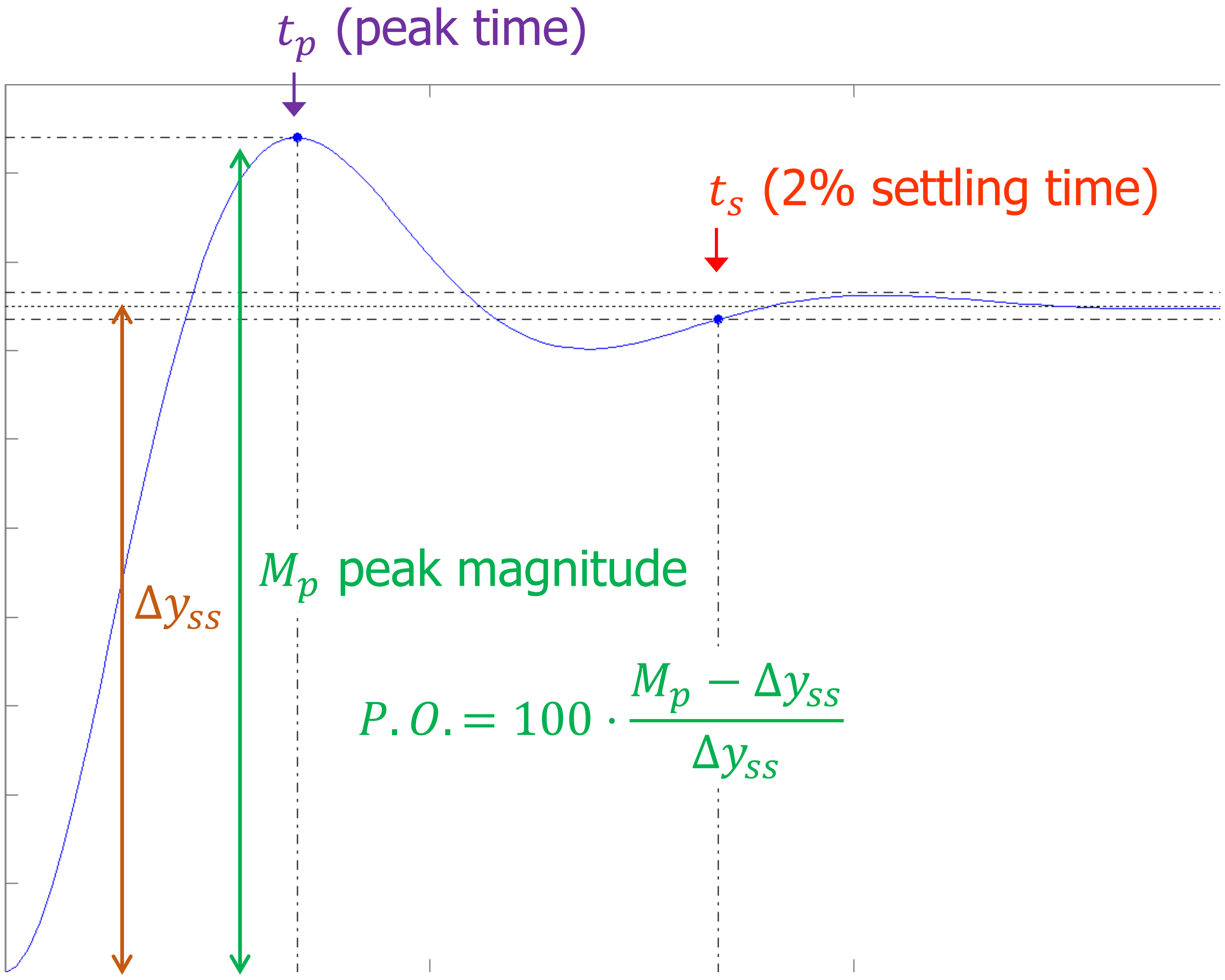
$$t_s \approx \frac{4}{\zeta \omega_n}$$

- The percent overshoot is:

$$P.O. = 100 \cdot \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

- The peak time is:

$$t_p = \frac{\pi}{\omega_d}$$



Second Order Systems—Examples

- Let x be the position of a vehicle and τ the torque of the motor. The transfer function $\frac{X(s)}{T(s)}$ corresponds, approximately, to a second order system.
- A block connected to a spring.
- An RLC circuit.

Second Order Systems—Example

Find the percent overshoot and the settling time of $H(s) = \frac{20}{4s^2+2s+1}$.

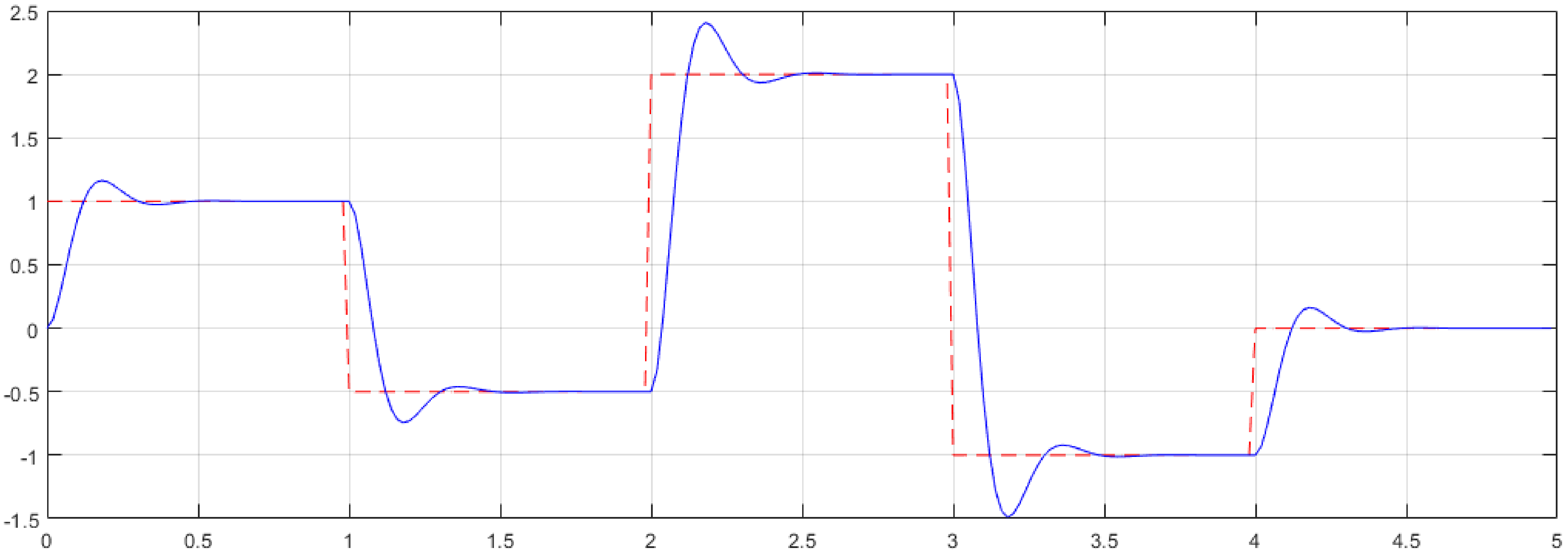
- First, let's write $H(s)$ in the form $\frac{A}{s^2+2\zeta\omega_n s+\omega_n^2}$.

$$\frac{20}{4s^2+2s+1} = \frac{20/4}{s^2+0.5s+0.25}$$

- We infer $2\zeta\omega_n = 0.5$ and $\omega_n^2 = 0.25$.
- Therefore, $\zeta = 0.5$ and $\omega_n = 0.5$ rad/s.
- Since $0 < \zeta < 1$, the following formulas may be used:

$$t_s \approx \frac{4}{\zeta\omega_n} = 16 \text{ sec and } P.O. = 100 \cdot \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 16.3\%$$

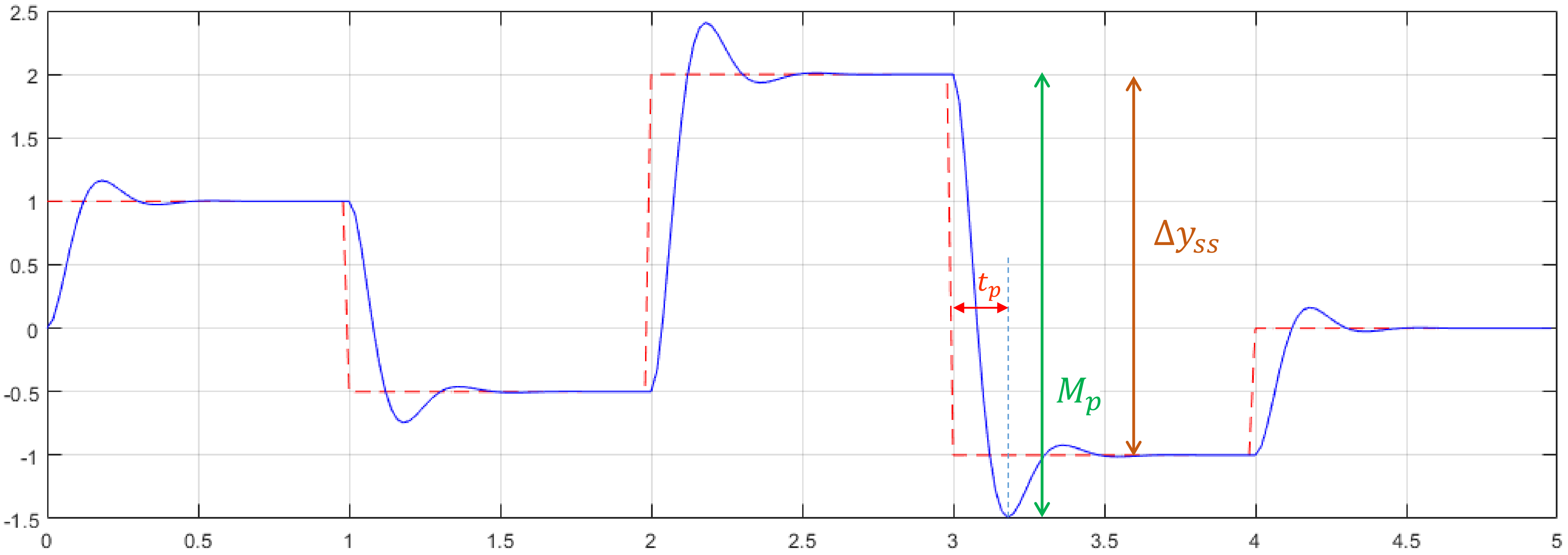
Second Order Systems—Example



Find ζ and ω_n.

Note that any of the five edges could be used to find ζ and ω_n.

Second Order Systems—Example

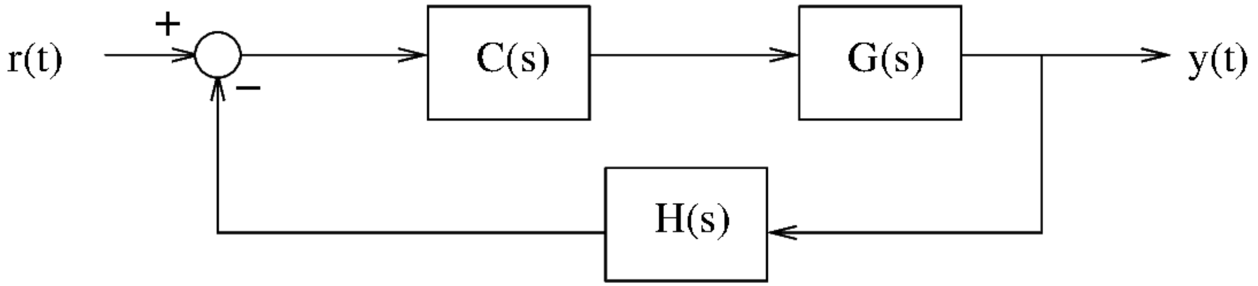


Since $\Delta y_{ss} = 3$ and $M_p = 3.5$, it follows that $P.O. = \frac{100}{6}$.

Now $P.O. \neq 0 \Rightarrow 0 < \zeta < 1$. Therefore, $P.O. = 100 \cdot \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$. This implies $\zeta = 0.495$.

From $t_p \simeq 0.2$ s and $t_p = \frac{\pi}{\omega_d}$ we find $\omega_d = 5\pi$ rad/s $\Rightarrow \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 18.1$ rad/s.

Second Order Systems—Example



Assume $G(s) = \frac{0.2}{s^2}$, $C(s) = k$, *and* $H(s) = 1 + k_d s$. *Find* k *and* k_d *so that the system responds to a step input with* $P.O. = 5\%$ *and* $t_s = 1$ *s.*

- Using the formulas

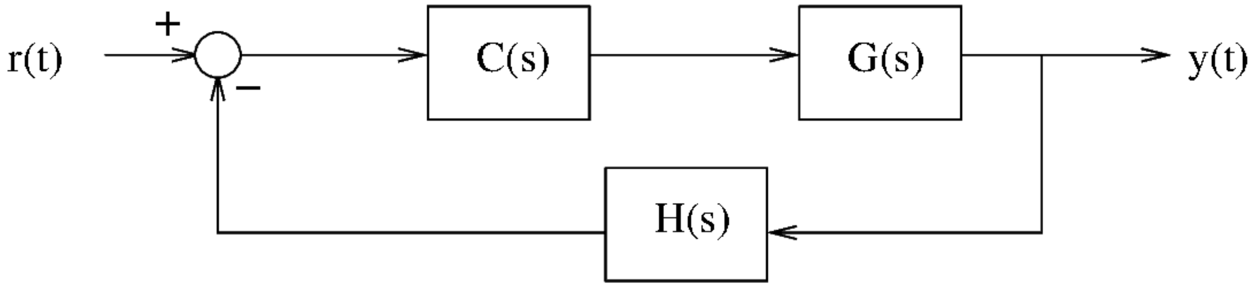
  $t_s \approx \frac{4}{\zeta\omega_n}$ and $P.O. = 100 \cdot \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

  we obtain $\zeta = 0.69$ and $\omega_n \approx \frac{4}{0.69}$ rad/s.

- This corresponds to second order system

$$\frac{A}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{A}{s^2 + 8s + 33.6}$$

Second Order Systems—Example



- The closed-loop transfer function is

$$\frac{Y}{R} = \frac{CG}{1 + CGH} = \frac{0.2k}{s^2 + 0.2kk_d s + 0.2k}$$

- It should equal

$$\frac{A}{s^2 + 8s + 33.6}$$

- It follows that $0.2kk_d = 8$ and $0.2k = 33.6$.
- Therefore, $k = 168$ and $k_d = 0.238$.