# Controller Implementation—Operational Amplifiers

Discrete Time Approximations. Operational Amplifiers.

M.V. Iordache, *EEGR3523 Mechatronics*, Spring 2020, LeTourneau University

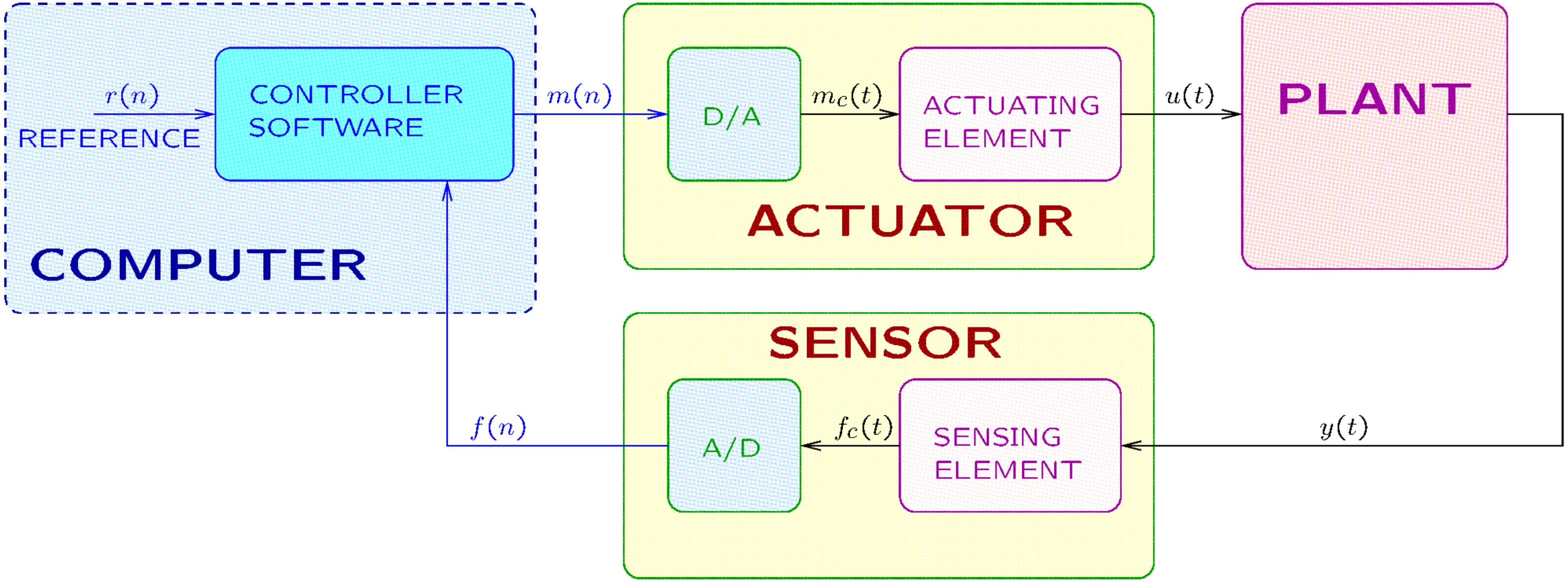
# Controller Implementation

- Control algorithms are commonly implemented on computers and microcontrollers.
  - This is a discrete-time implementation.
- Analog implementations can be done with analog electronic components such as *operational amplifiers*.
  - This is a continuous-time implementation of the control algorithms.

# Discrete-Time Implementation

The control system will include:

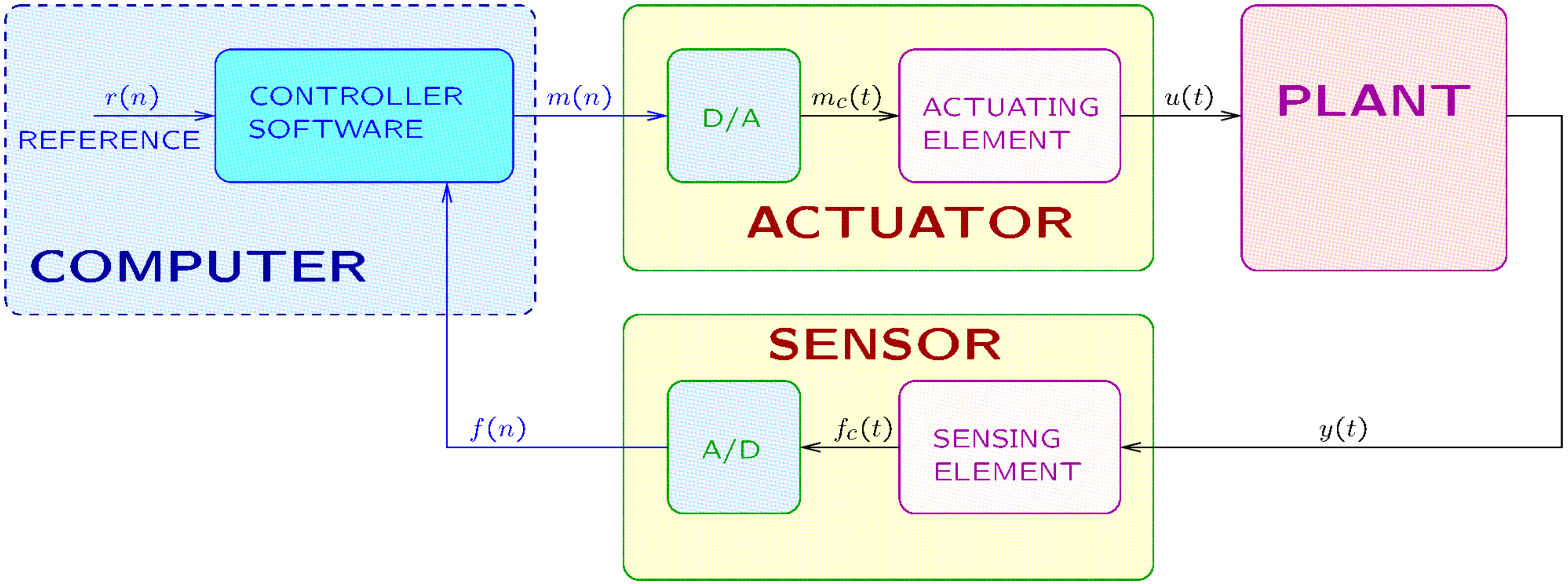
- Digital to analog converters (D/A).
- Analog to digital controllers (A/D).
- Software implementing the control algorithm.



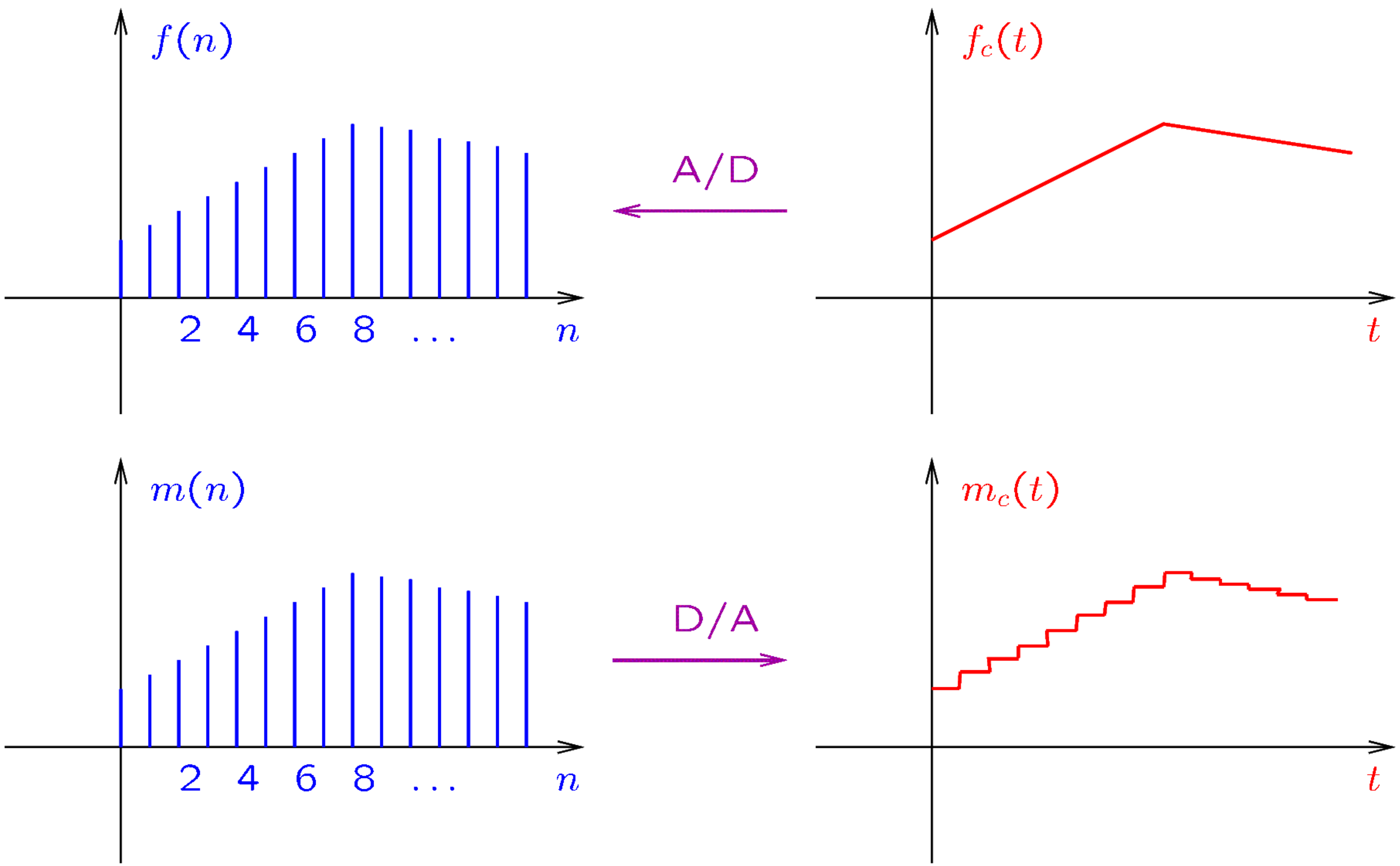
# Discrete-Time Implementation

The control system involves:

- Continuous-time signals: $m_c(t), u(t), y(t), f_c(t)$, where $t$ is the time.
- Discrete-time signals: $r(n), m(n), f(n)$, where $n = 0, 1, 2, \ldots$ is an integer.



# Discrete-Time Implementation




- The A/D samples periodically the input.
- Let $T$ be the sampling interval.

$$f(n) = f_c(nT)$$

- The D/A outputs the latest discrete-time value:

$$m_c(t) = m(n) \text{ when } nT \leq t < (n+1)T$$

- The D/A output will change in steps.

# Discrete-Time Implementation

- Suppose that the following PID control law should be implemented in discrete time.

$$u_c(t) = k_p \cdot e_c(t) + k_i \int_0^t e_c(x)dx + k_d \cdot \dot{e}_c(t)$$

- The discrete-time controller should output $u(n)$ so that

$$u_c(t) = u(n) \text{ when } nT \leq t < (n+1)T$$

- Moreover, the input of the controller should equal the samples of $e_c(t)$:

$$e(n) = e_c(nT)$$

- A possible way to implement the controller is by approximating the derivative and the integral.

$$\dot{e}_c(nT) \simeq \frac{e(n) - e(n-1)}{T}$$

$$\int_0^{nT} e_c(x)dx \simeq \int_0^{(n-1)T} e_c(x)dx + T \cdot e(n-1)$$

# Discrete-Time Implementation

- It follows that the discrete-time controller will be described by the equations:

$$u(n) = k_p e(n) + k_i v(n) + k_d \frac{e(n) - e(n-1)}{T}$$

$$v(n) = v(n-1) + T \cdot e(n-1)$$

where $v(n)$ approximates the integral.

- *Note that the discrete-time control law depends on the sampling interval T.*

*Example: Find a discrete-time controller implementation of the continuous-time PID controller $C(s) = 300 + \frac{0.4}{s} + 0.2s$. Assume $T = 10\ ms$.*

- *Note that* $k_p = 300$, $k_i = 0.4$, *and* $k_d = 0.2$.

$$u(n) = 300e(n) + 0.4v(n) + 0.2\frac{e(n) - e(n-1)}{0.01}$$

$$= 320e(n) + 20e(n-1) + 0.4v(n)$$

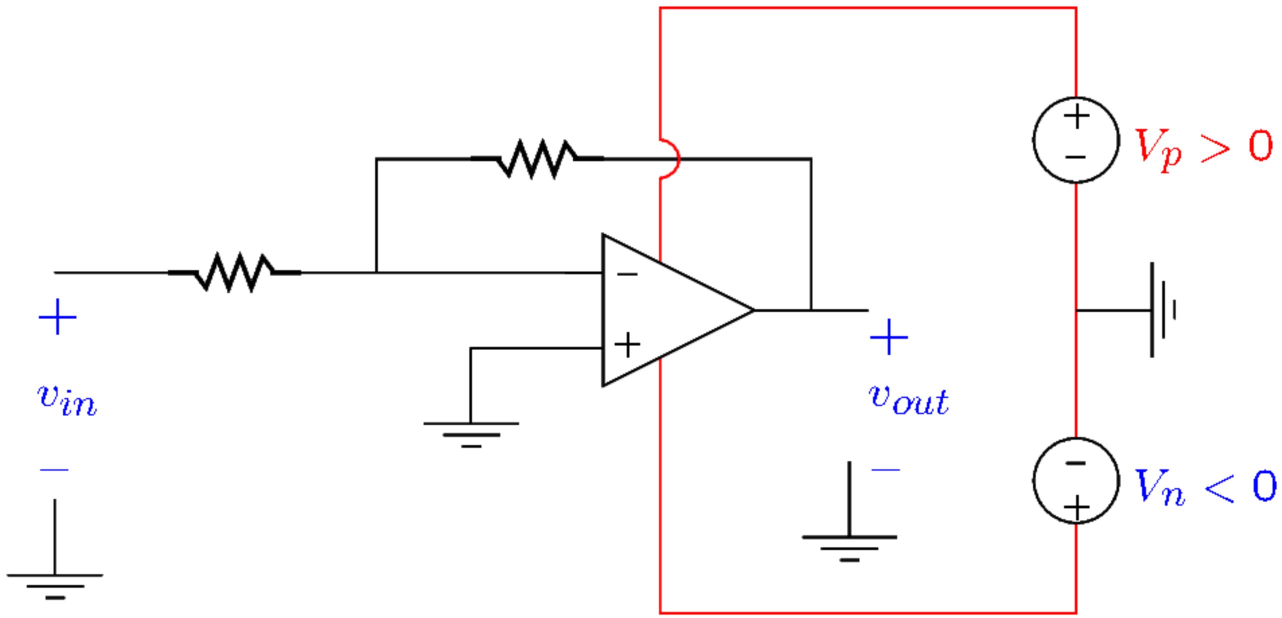
$$v(n) = v(n-1) + 0.01 \cdot e(n-1)$$

# Continuous-Time Implementation

- A possible way to obtain a continuous-time implementation is using operational amplifiers.
- Operational amplifiers are widely used in electronic circuits.
- They allow precise implementations of gains and transfer functions.
  - In particular, they could be used to implement a PID control law.
  - They have been used in analog computers, which are devices simulating precisely solutions of differential equations.

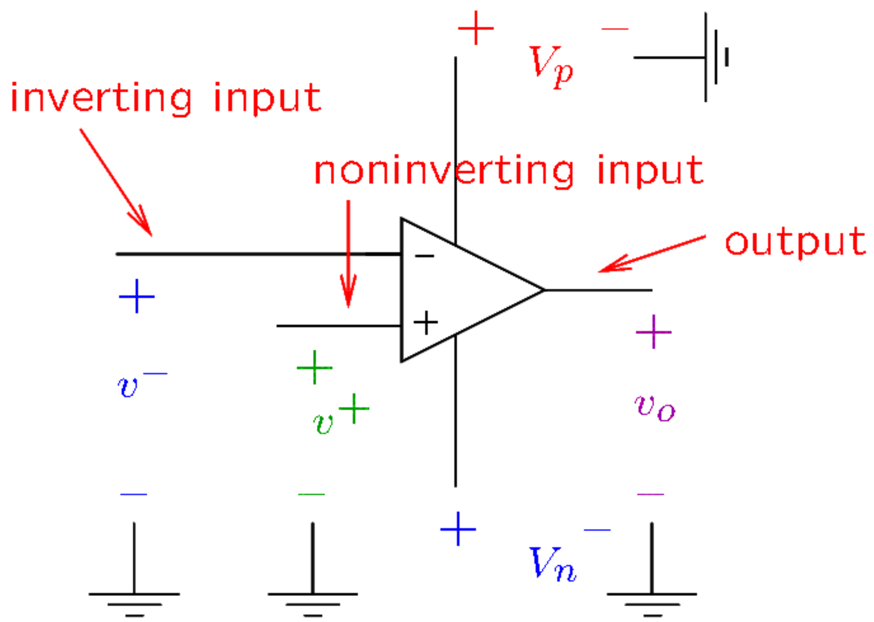
# Operational Amplifiers

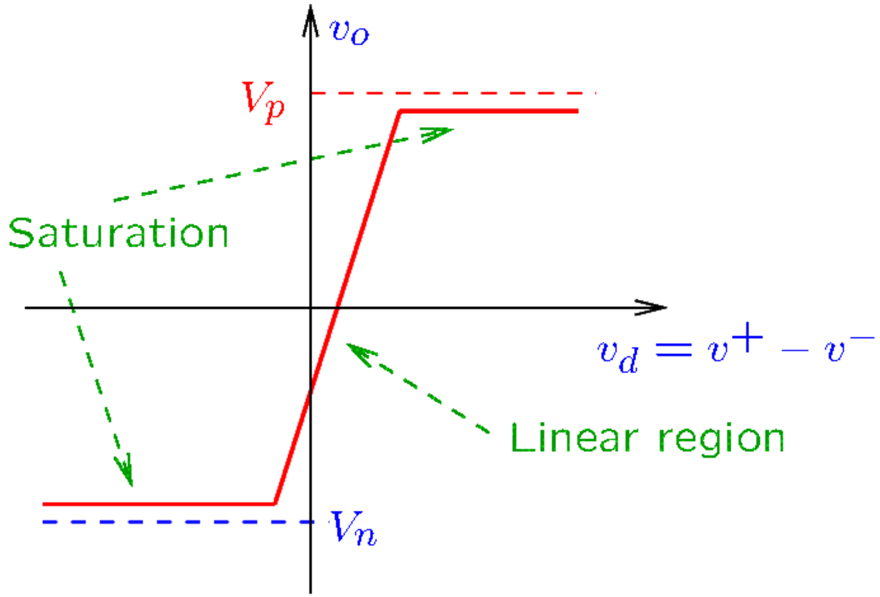
- The figure shows an amplifier circuit that uses an operational amplifier.
- Operational amplifiers are commonly used to amplify signals.
- They have pins for the input and output signals (shown in black) and supply pins (shown in red) used to power the operational amplifier.



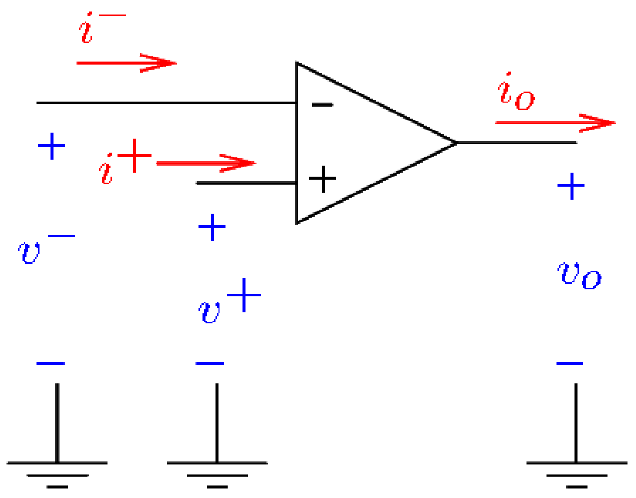
# Operational Amplifiers

- Let $v^+$ be the *noninverting input* voltage and $v^-$ the *inverting input* voltage.
- Let $V_p$ be the voltage on the positive supply pin and $V_n$ the voltage on the negative supply pin.
- In *saturation*, the output voltage is $v_o \simeq V_p$ or $v_o \simeq V_n$.
- In the *linear region*, $v_o \simeq A(v^+ - v^-)$, where $A$ is the operational amplifier gain.



# Operational Amplifiers




- Because the gain is typically very large:

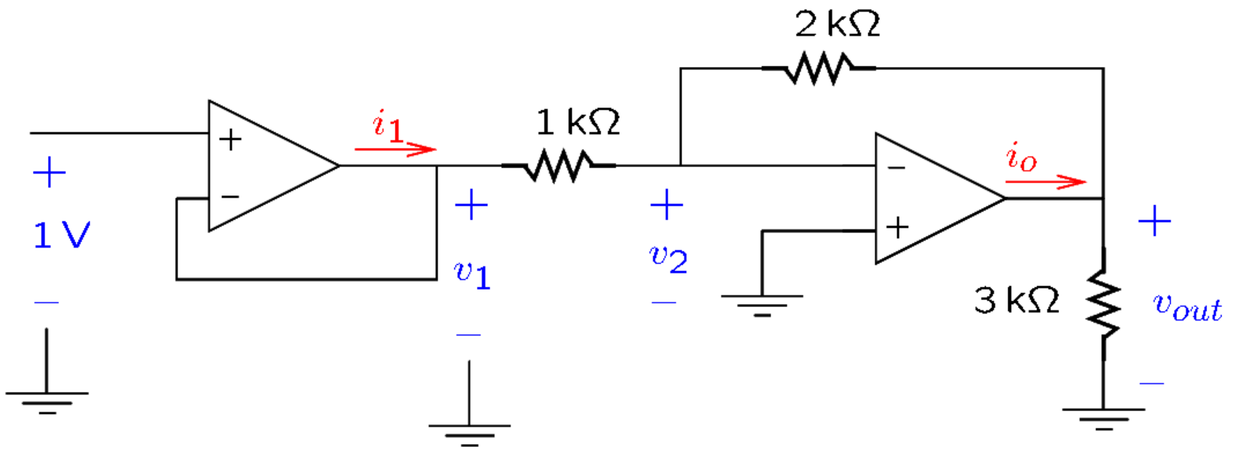
$$v^+ \simeq v^- \text{ in the linear region.}$$

- The input currents are normally negligible:

$$i^+ \simeq i^- \simeq 0.$$

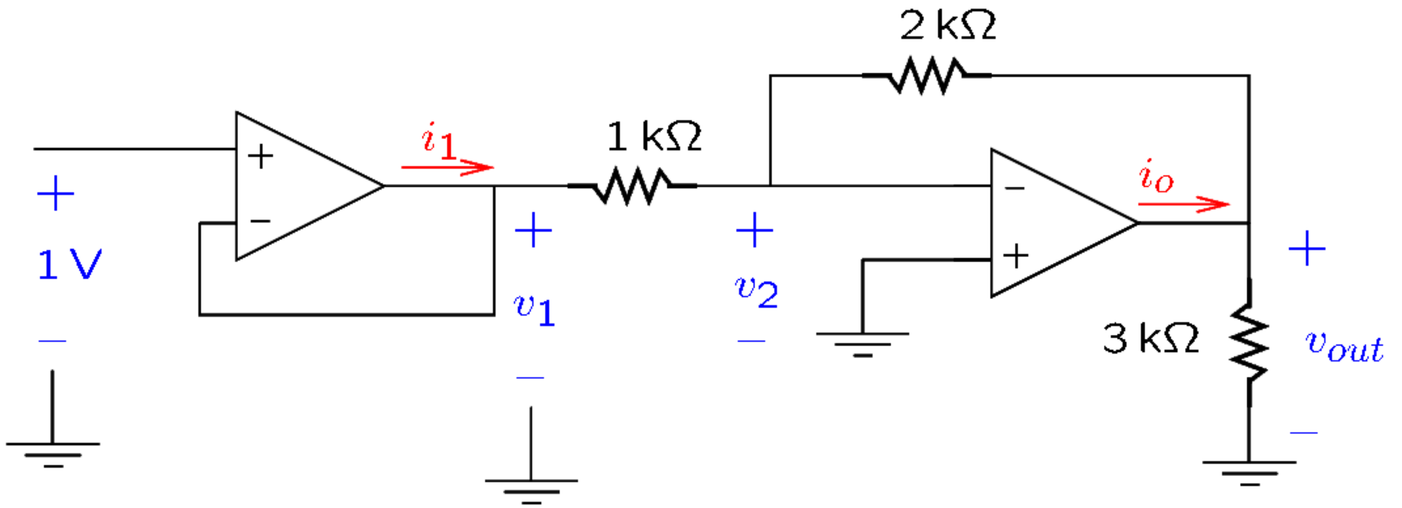
- The equations $v^+ = v^-$ and $i^+ = i^- = 0$ can be used to solve operational amplifier circuits.

*Example: Find the unknown voltages and currents. Assume linear region operation.*



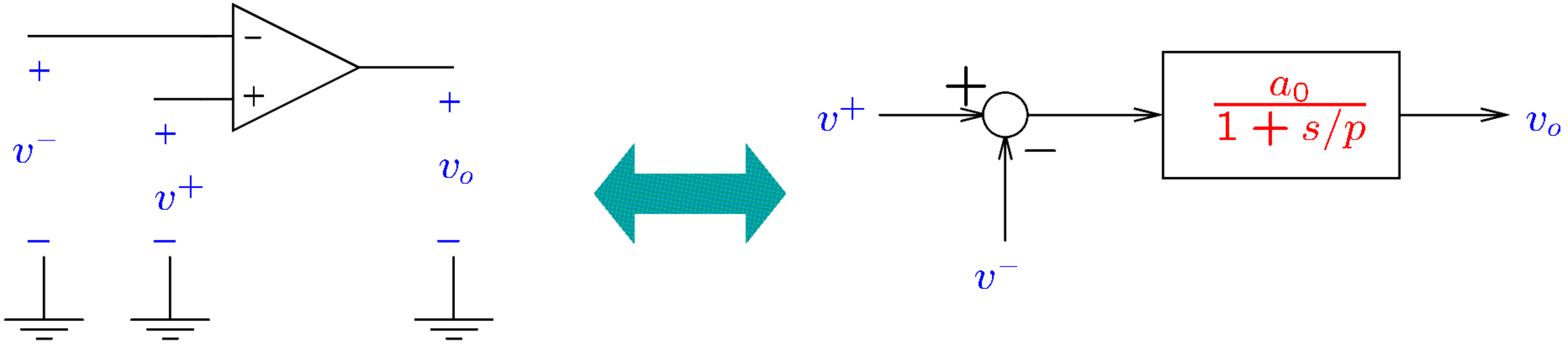
# Operational Amplifiers—Example

- For the first amplifier, $v^+ = v^- \Rightarrow v_1 = 1\ V$.
- For the second amplifier, $v^+ = v^- \Rightarrow v_2 = 0\ V$.
- Writing KCL at the node of $v_1$: $i_1 = i^- + \frac{v_1 - v_2}{1\ k\Omega} = 0 + \frac{1-0}{1\ k\Omega} = 1\ mA$.
- Writing KCL at the node of $v_2$: $\frac{v_1 - v_2}{1\ k\Omega} + \frac{v_{out} - v_2}{2\ k\Omega} = i^- = 0$. Therefore, $v_{out} = -2\ V$.
- Writing KCL at the node of $v_{out}$: $i_o = \frac{v_{out}}{3\ k\Omega} + \frac{v_{out} - v_2}{2\ k\Omega} = -1.67\ mA$.



# Operational Amplifiers—Stability

- Circuits with operational amplifiers, if improperly designed, *can be unstable*.
- To understand instability, a first order approximation could be used for the transfer function.
  - The pole of the transfer function is often in the range $-10$ to $-100\ rad/s$.
- The approximation refers to the linear region; $a_0$ is the DC gain of the amplifier in the linear region.



# Operational Amplifiers—Stability Example

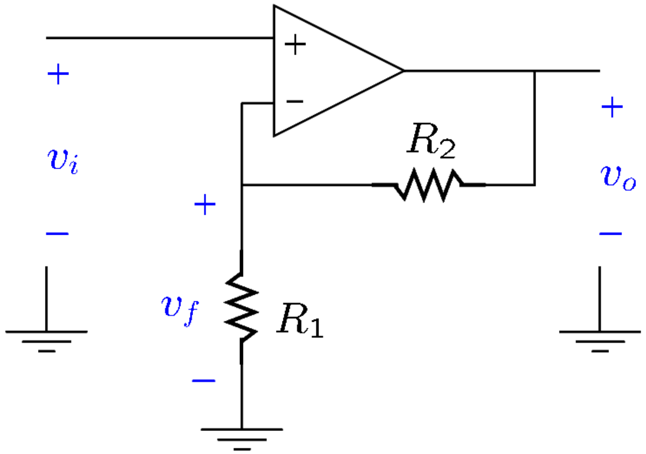
*Example: The circuit and the block diagram below are equivalent.*

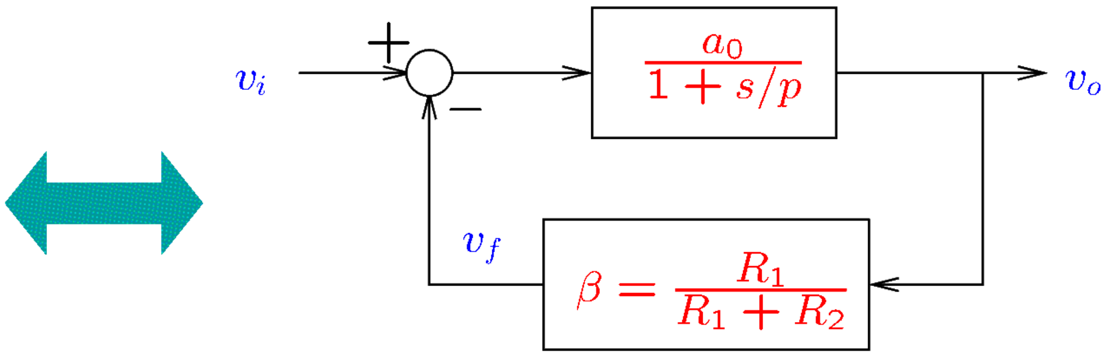
$$V_f(s) = \beta V_o(s)$$

$$V_o(s) = \frac{a_0}{1 + \frac{s}{p}}\left(V_i(s) - V_f(s)\right)$$

*The closed-loop transfer function is:*

$$CL(s) = \frac{a_0}{1 + a_0\beta + s/p}$$



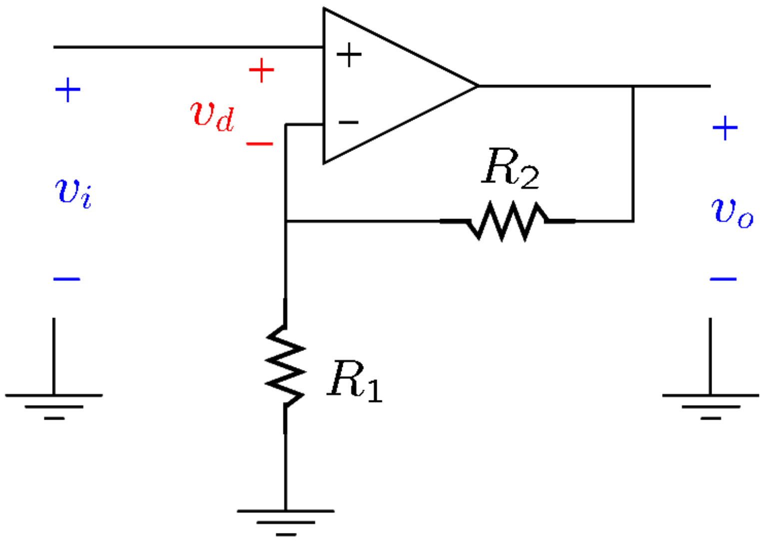
# Operational Amplifiers—Stability Example

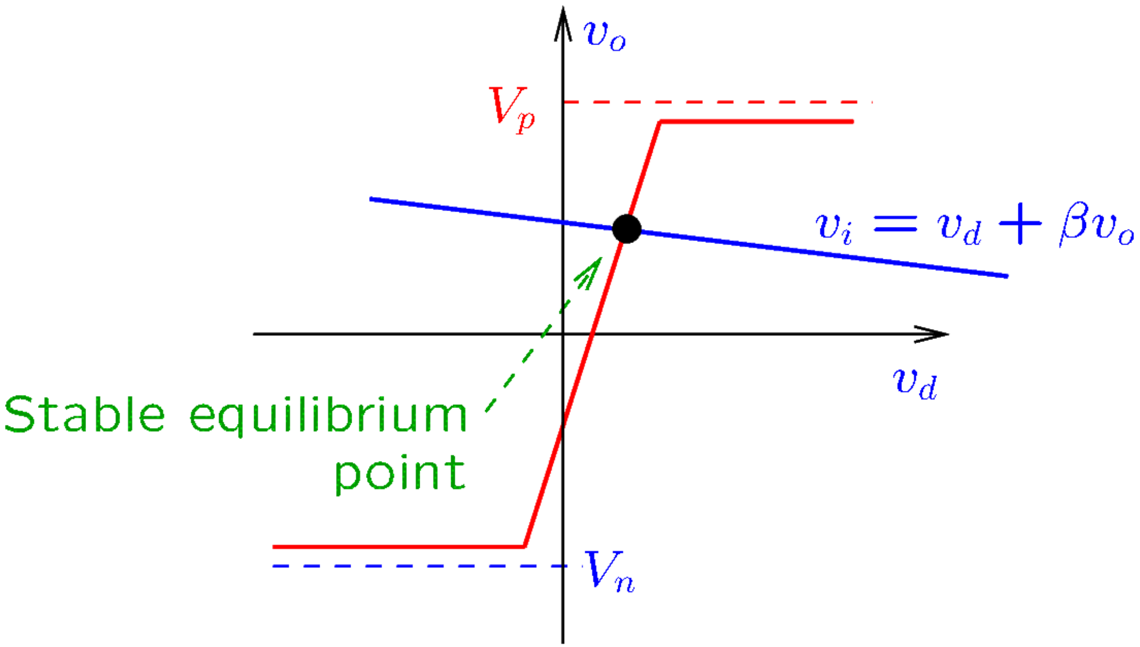
*The closed-loop transfer function is:*

$$CL(s) = \frac{a_0}{1 + a_0\beta + s/p}$$

*Since $p > 0$, the pole of the closed-loop transfer function is in the LHP.*

*Therefore, the system is stable!*



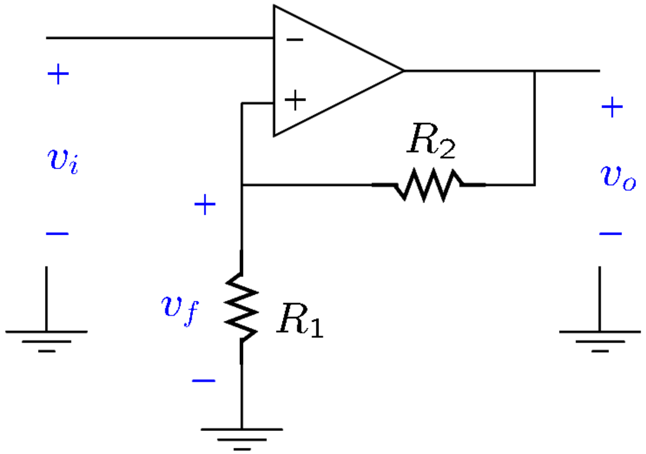
# Operational Amplifiers—Stability Example 2

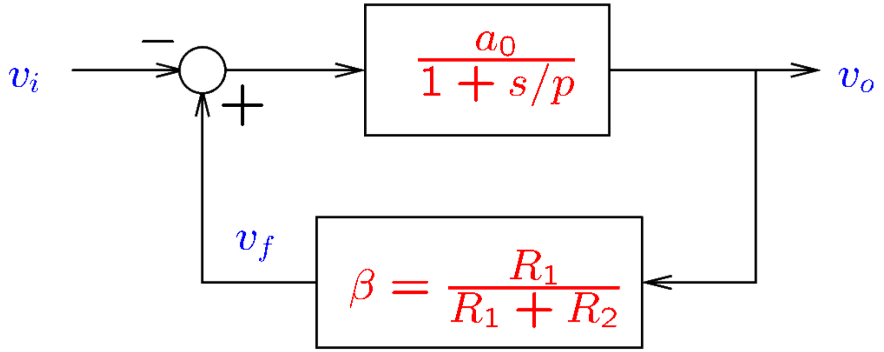
*Example: Consider the previous circuit after interchanging the connections of the inputs of the operational amplifier.*

*The closed-loop transfer function is now:*

$$CL(s) = \frac{-a_0}{1 - a_0\beta + s/p}$$

*The pole is at* $p(a_0\beta - 1)$. *Unless* $a_0\beta < 1$, *which would be quite unusual, the pole is in the RHP and the circuit is unstable in the linear region.*



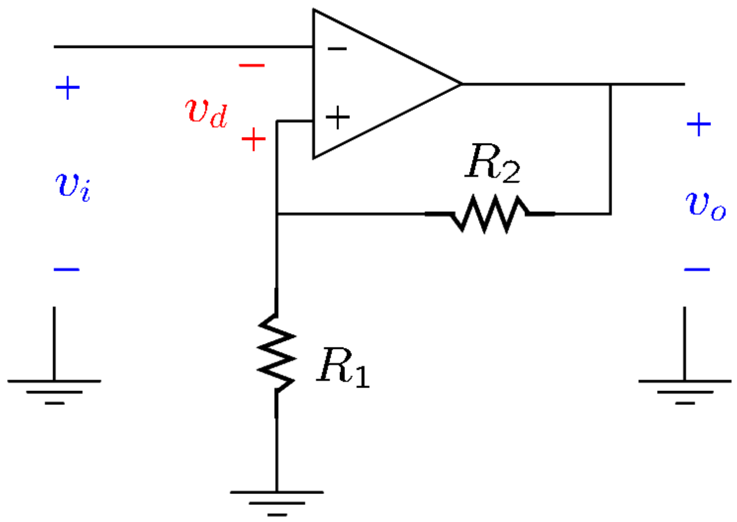



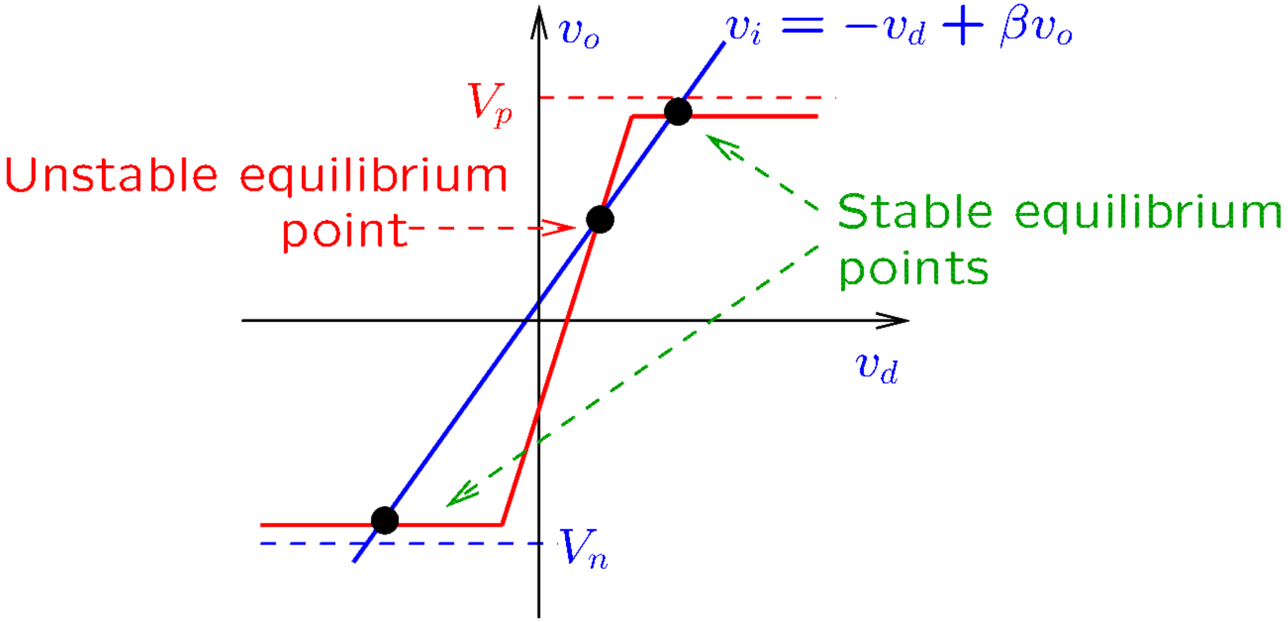
# Operational Amplifiers—Stability Example 2

*The circuit has three equilibrium points.*

*The equilibrium point in the linear region is unstable, as we have proved.*

*Therefore, the circuit will operate in one of the other two equilibrium points, outside of the linear region.*



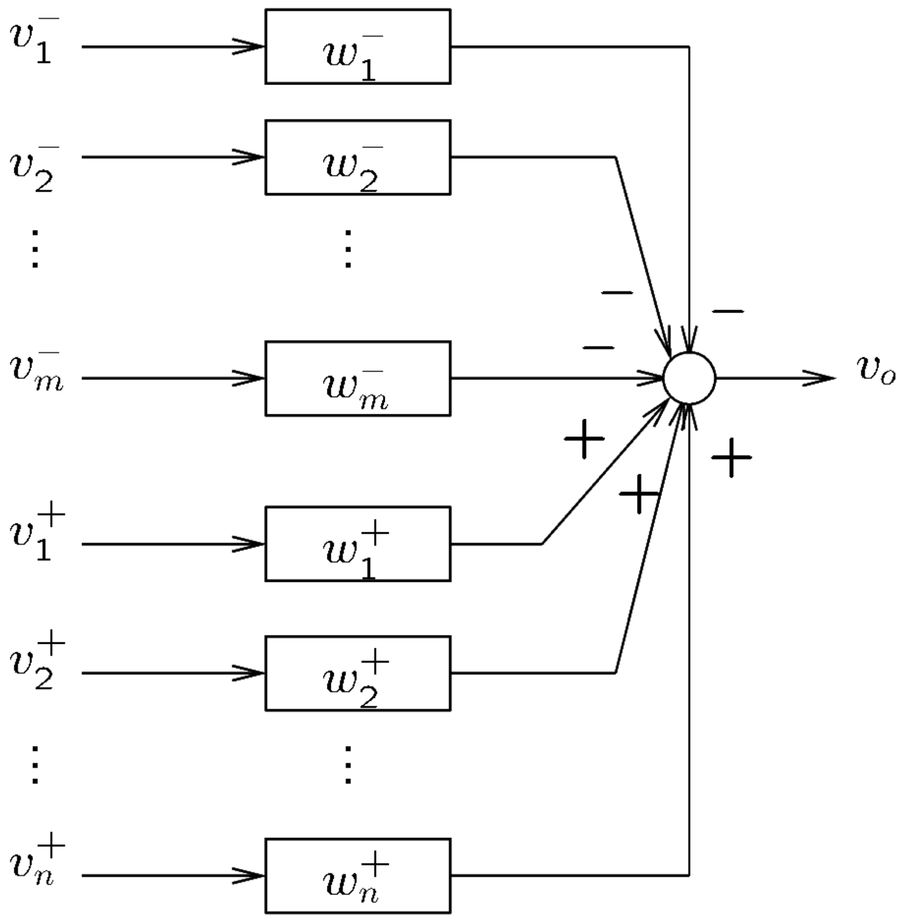



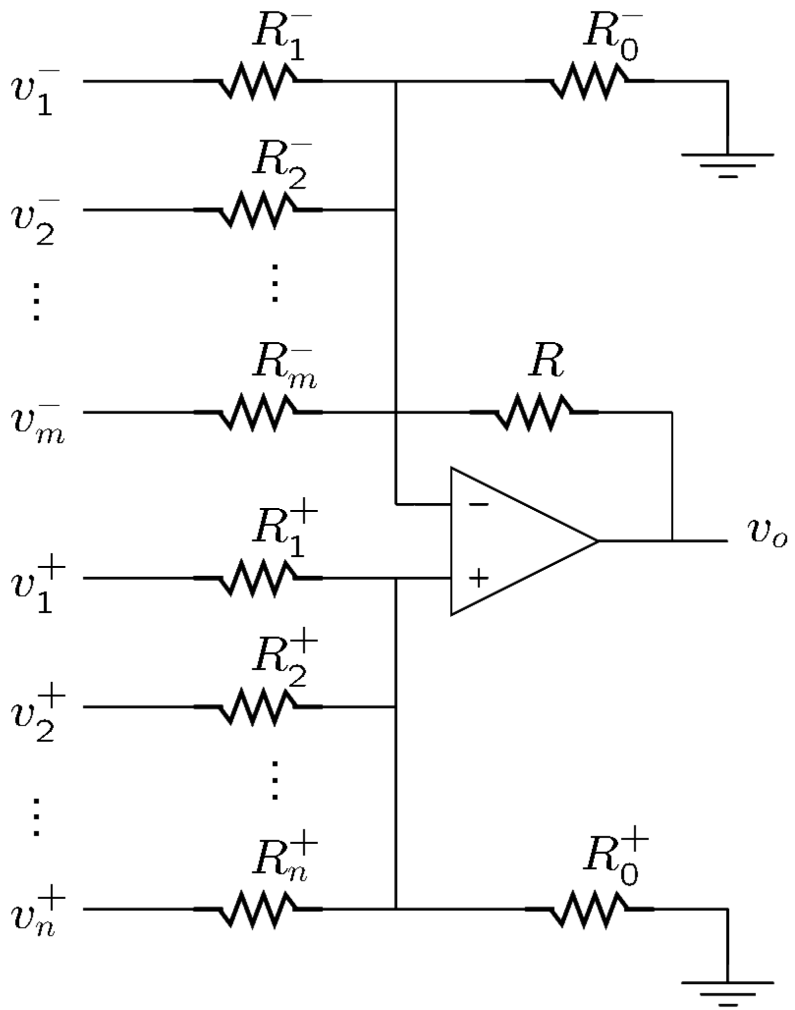
# Block Diagrams and Operational Amplifiers

- Consider a block diagram representing a control algorithm.
- Suppose that the algorithm should be implemented in hardware.
- Various blocks can be implemented with operational amplifiers, including
  - Summing junctions
  - Transfer function blocks
- See the *operational amplifier handout*.
- For a more efficient implementation, several blocks could be combined and implemented with a single operational amplifier.

# Summing Junctions

An operational amplifier can implement the weighted sum $v_o = \sum_i^n w_i^+ v_i^+ - \sum_i^m w_i^- v_i^-$.



# Summing Junctions

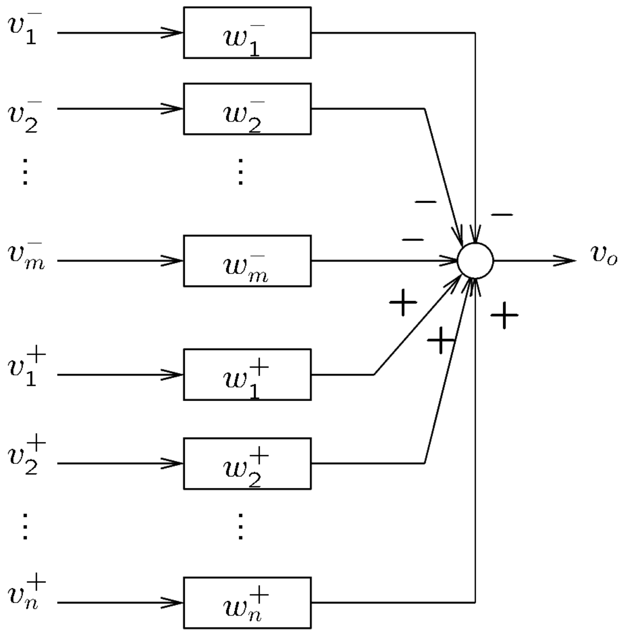
- The weights $w_1^+ \ldots w_n^+$ and $w_1^- \ldots w_m^-$ are given.
- Select $R_1^+ \ldots R_n^+$ and $R_1^- \ldots R_m^-$ so that

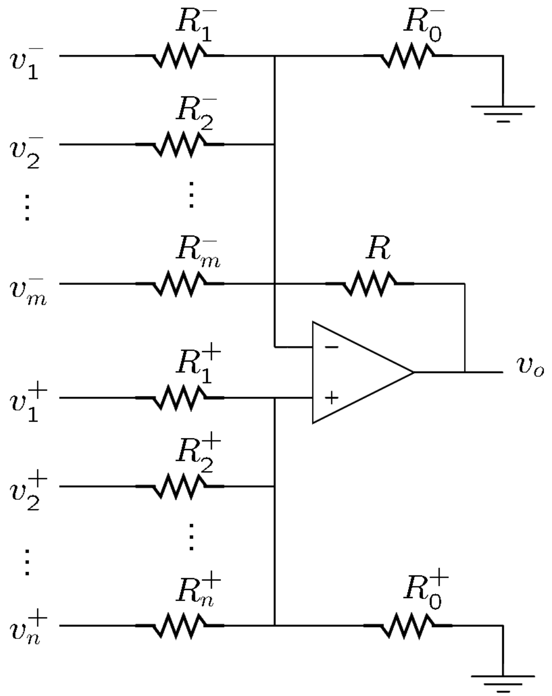
$$w_i^+ = \frac{R}{R_i^+} \text{ and } w_i^- = \frac{R}{R_i^-}.$$

- Select $R_0^+$ and $R_0^-$ so that

$$1 + \sum_{i=0}^{m} \frac{R}{R_i^-} = \sum_{i=0}^{n} \frac{R}{R_i^+}.$$

- *Note that choosing $R_0 = \infty$ is fine (this means open circuit—no resistor).*
- *The proof follows immediately via nodal analysis.*



# Summing Junctions—Example

*Find a circuit implementing* $v_0 = 3v_1 - 2v_2 - 5v_3$.

- *There is one "+" input, so* $n = 1$.
- *There are two "−" inputs, so* $m = 2$.
- *The constraints for* $R_1^+, R_1^-, R_2^-$ *are:*

  $$\frac{R}{R_1^+} = 3 \Rightarrow R_1^+ = \frac{R}{3}; \frac{R}{R_1^-} = 2 \Rightarrow R_1^- = \frac{R}{2}; \frac{R}{R_2^-} = 5 \Rightarrow R_2^- = \frac{R}{5}.$$

- *The constraints for* $R_0^+$ *and* $R_0^-$ *are:*

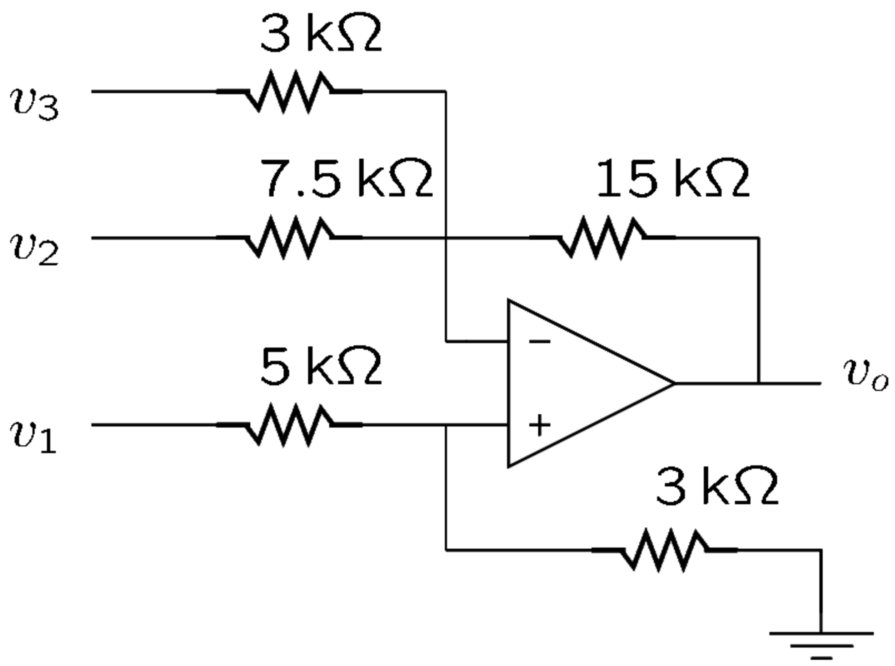
  $$1 + \frac{R}{R_0^-} + \frac{R}{R_1^-} + \frac{R}{R_2^-} = \frac{R}{R_0^+} + \frac{R}{R_1^+} \Rightarrow 1 + \frac{R}{R_0^-} + 2 + 5 = \frac{R}{R_0^+} + 3.$$

- The equation has many possible solutions; the simplest is

  $R_0^+ = \frac{R}{5}$ and $R_0^- = \infty$ (no resistor)

# Summing Junctions—Example

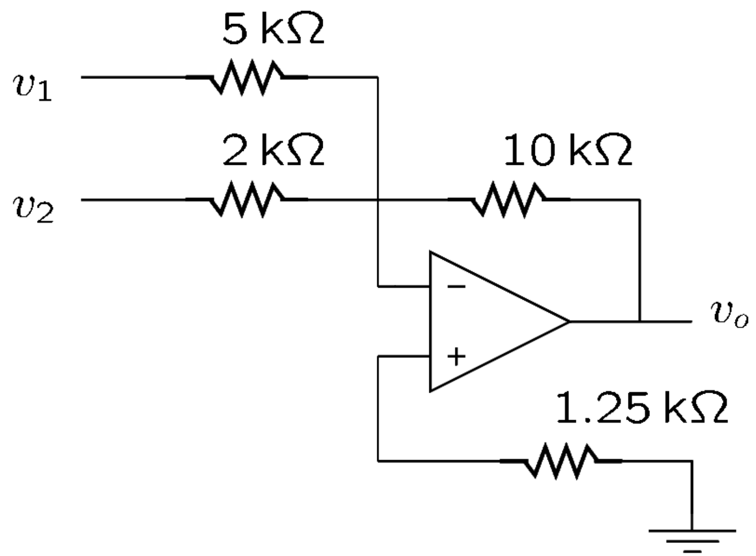
- *Let* $R = 15\ k\Omega$ *(this is an arbitrary choice).*
- *Then* $R_2^- = R_0^+ = 3\ k\Omega$, $R_1^- = 7.5\ k\Omega$, *and* $R_1^+ = 5\ k\Omega$.
- *Potentiometers (variable resistors) may be needed to adjust precisely the weights of each input.*



# Summing Junctions—Example 2

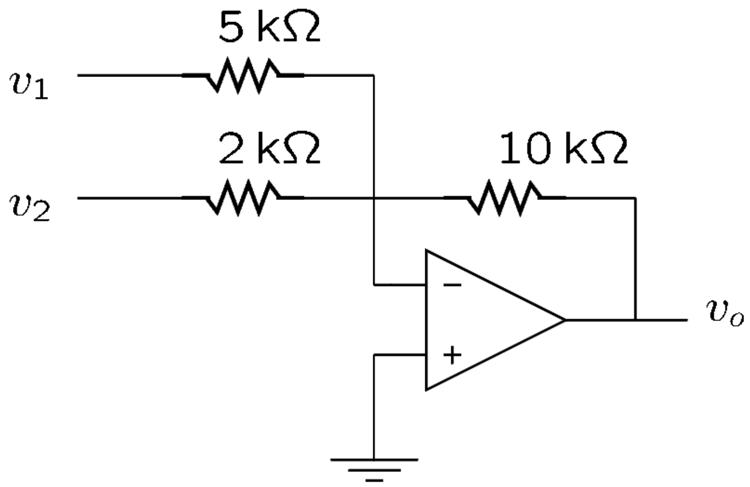
*Find a circuit implementing* $v_0 = -2v_1 - 5v_2$.

- *There are no "+" inputs, therefore* $n = 0$.
- *There are 2 "−" inputs, therefore* $m = 2$.
- $R_1^- = \frac{R}{2}$ *and* $R_2^- = \frac{R}{5}$.
- *The constraint* $1 + \frac{R}{R_0^-} + \frac{R}{R_1^-} + \frac{R}{R_2^-} = \frac{R}{R_0^+}$ *implies* $1 + \frac{R}{R_0^-} + 2 + 5 = \frac{R}{R_0^+}$, *for which we could choose* $R_0^- = \infty$ *and* $R_0^+ = \frac{R}{8}$.
- *The figure shows an implementation with* $R = 10\ k\Omega$.



# Summing Junctions—Remarks

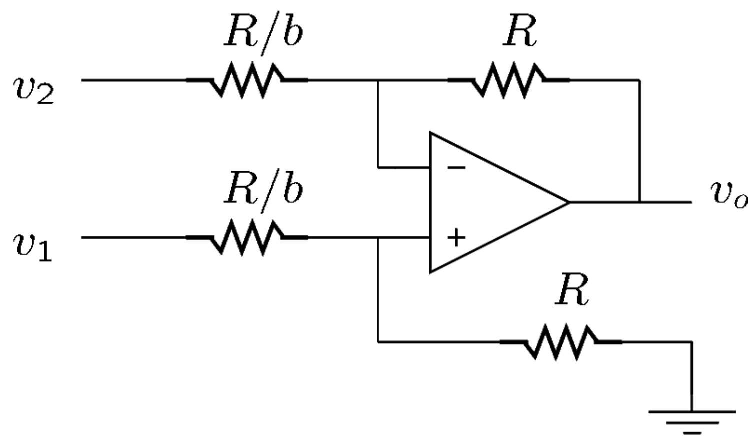
- *The method presented here does not lead to unique solutions.*
- *For example, the circuit shown here is also a solution of the previous example.*
- *However, the solution we have obtained earlier is better, as it can reduce the effect of the bias currents $i^-$ and $i^+$, which we normally neglect.*



# Summing Junctions—Example 3

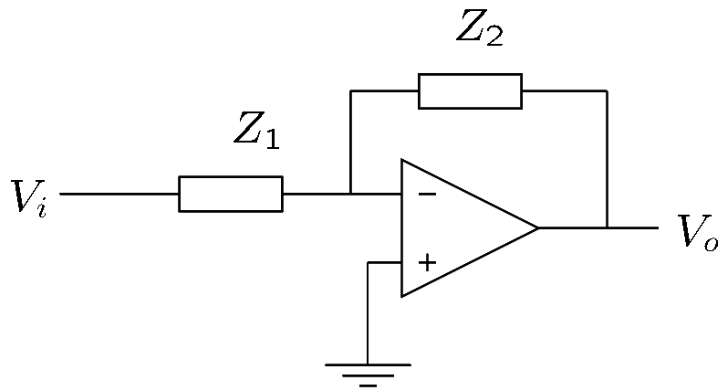
*Find a circuit implementing $v_0 = bv_1 - bv_2$, where $b$ is a constant.*

- *There is one "+" input, so $n = 1$.*
- *There is one "−" input, so $m = 1$.*
- $R_1^- = \frac{R}{b}$ and $R_1^+ = \frac{R}{b}$.
- *The constraint $1 + \frac{R}{R_0^-} + \frac{R}{R_1^-} = \frac{R}{R_0^+} + \frac{R}{R_1^+}$ implies $1 + \frac{R}{R_0^-} + b = \frac{R}{R_0^+} + b$, for which we could choose $R_0^- = \infty$ and $R_0^+ = R$.*
- *The figure shows the circuit.*



# Transfer Functions

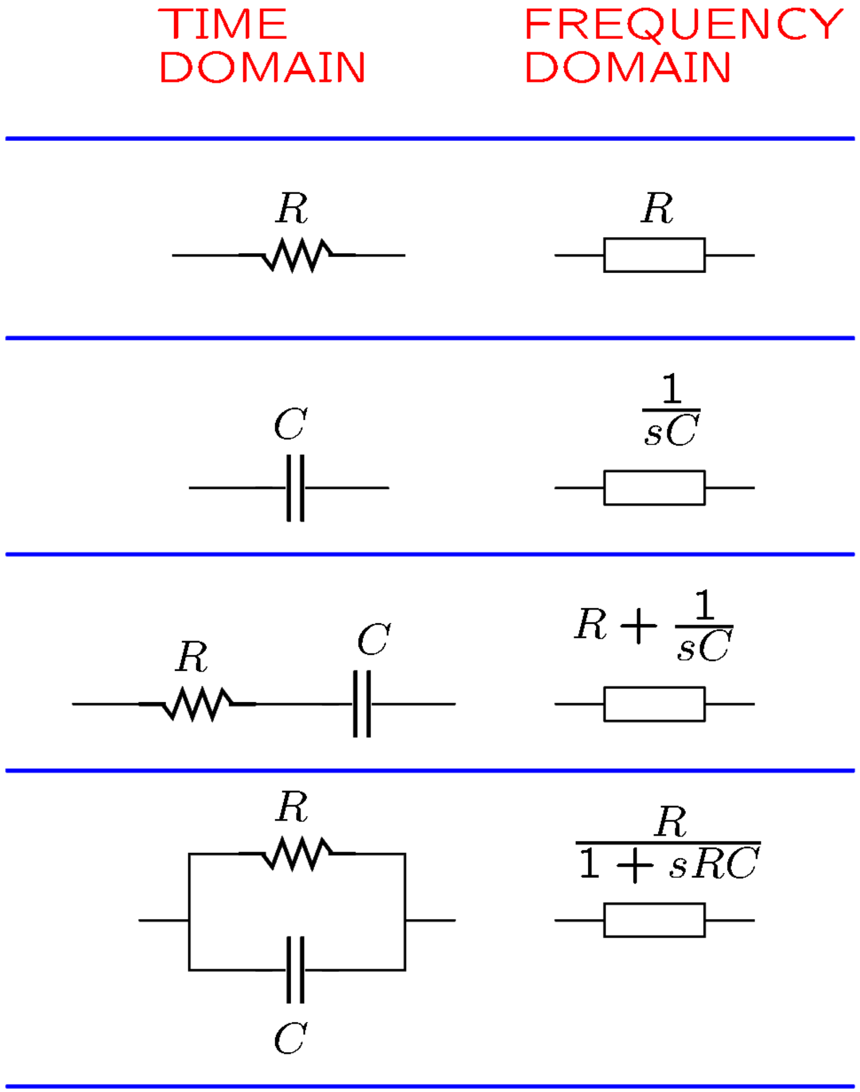
- Transfer functions can be implemented with resistors, capacitors, and *the inverting amplifier*:




- The circuit implements the transfer function

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1}$$

- Each of $Z_i$ and $Z_f$ can be a capacitor, resistor, or a combination of the two.
- This solution is not unique; there are other ways to implement a transfer function.



# Transfer Functions—Example

*Implement the transfer function* $-\frac{100}{s+10}$.

- *Let* $Z_1 = R_1$ *and* $Z_2 = \frac{R_2}{1+sR_2C_2}$.
- *The transfer function is*

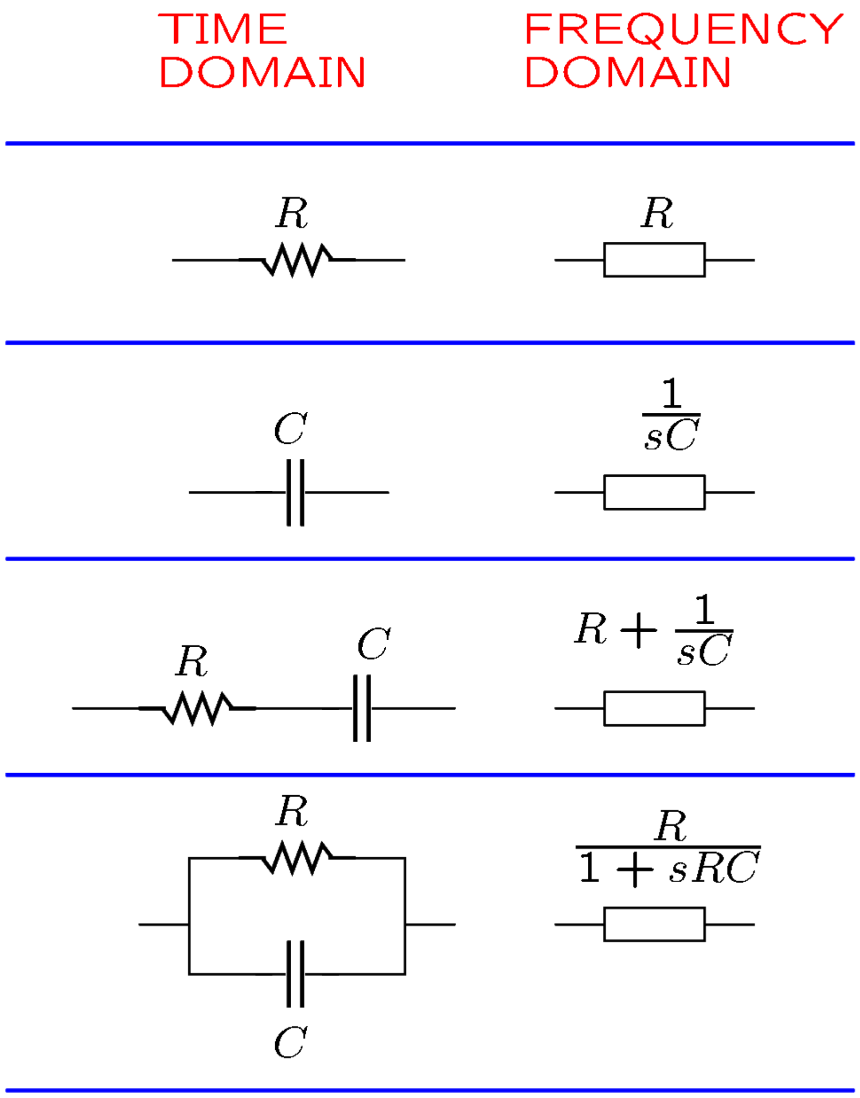
$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2/R_1}{1+sR_2C_2}$$

- *Let us rewrite the given transfer function in a more convenient form:*

$$-\frac{100}{s+10} = -\frac{10}{1+0.1s}$$

- *We have the constraints*

$$\frac{R_2}{R_1} = 10 \text{ and } R_2C_2 = 0.1.$$



# Transfer Functions—Example

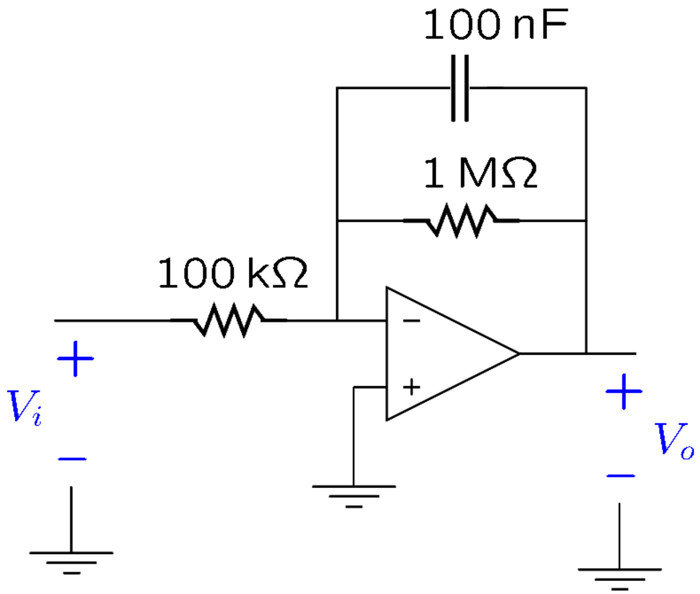
- *We have the constraints*

  $\frac{R_2}{R_1} = 10$ *and* $R_2 C_2 = 0.1$.

- *We can choose (arbitrarily) the value of one variable and then solve for the other two.*

- *Let* $R_2 = 1\ M\Omega$.

  $\Rightarrow R_1 = 100\ k\Omega$ *and* $C_2 = 100\ nF$.



# Transfer Functions—Example 2

*Implement the transfer function* $-100 \cdot \frac{s+10}{s+100}$.

- *Let* $Z_1 = \frac{R_1}{1+sR_1C_1}$ *and* $Z_2 = \frac{R_2}{1+sR_2C_2}$.
- *The transfer function is*

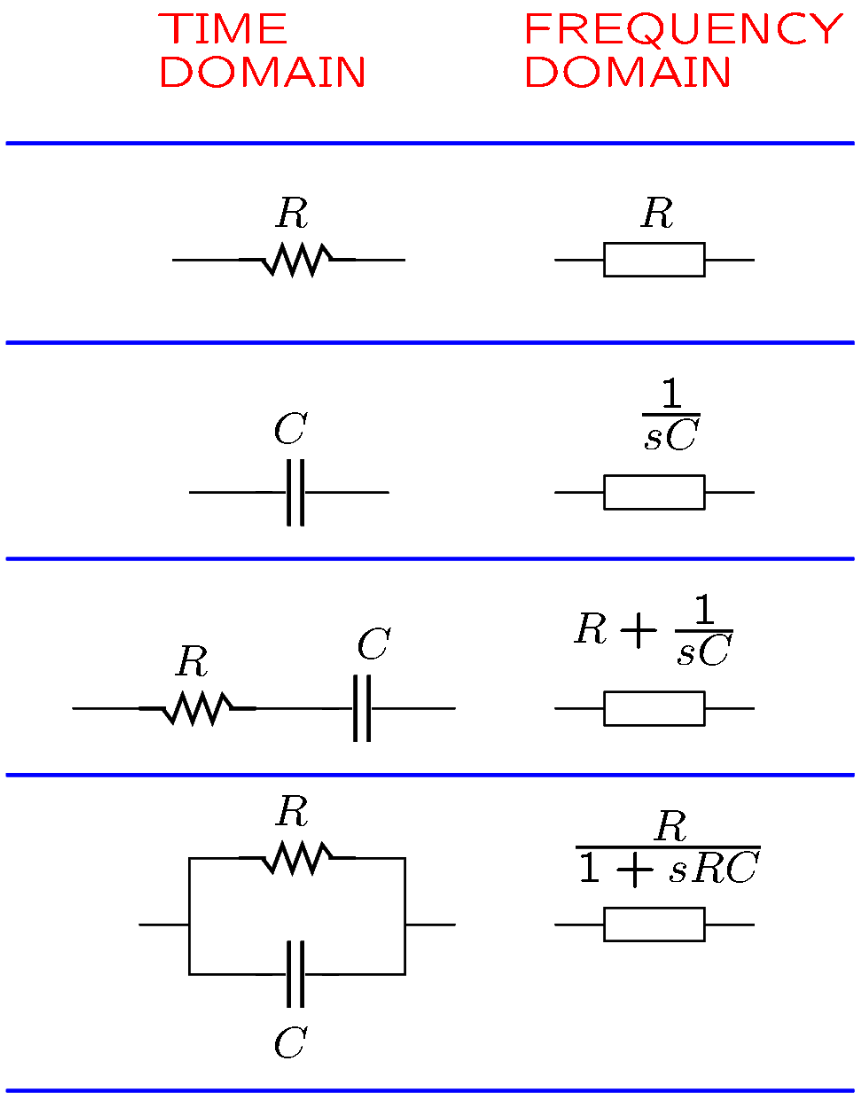
$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2}{R_1} \cdot \frac{1+sR_1C_1}{1+sR_2C_2}$$

- *Let us rewrite the given transfer function in a more convenient form:*

$$-100 \cdot \frac{s+10}{s+100} = -10\,\frac{1+0.1s}{1+0.01s}$$

- *We have the constraints*

$\frac{R_2}{R_1} = 10$ *and* $R_2C_2 = 0.01$ *and* $R_1C_1 = 0.1$.



# Transfer Functions—Example 2

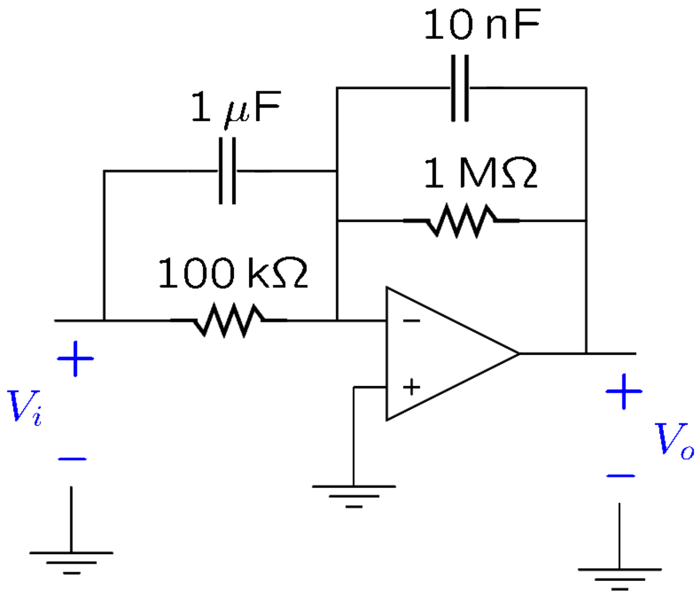
- *We have the constraints*

  $\frac{R_2}{R_1} = 10$ *and* $R_2C_2 = 0.01$ *and* $R_1C_1 = 0.1$.

- *We can choose (arbitrarily) the value of one variable and then solve for the other two.*

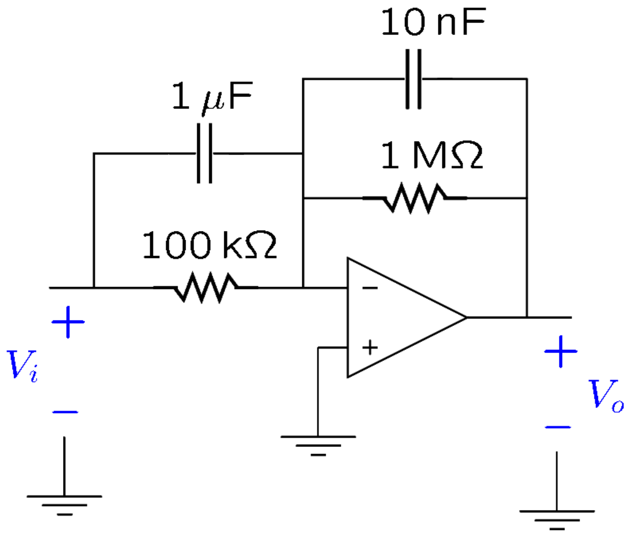
- *Let* $R_2 = 1\ M\Omega$.

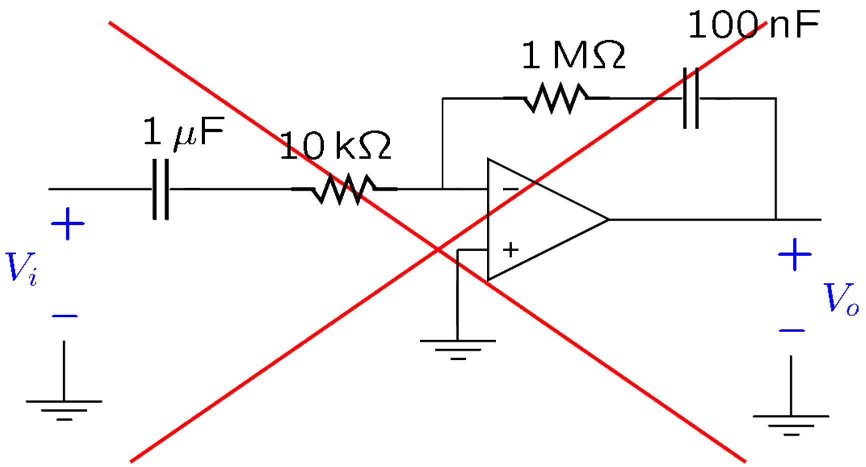
$$\Rightarrow C_2 = 10\ nF \text{ and } R_1 = 100\ k\Omega$$
$$\Rightarrow C_1 = 1\ \mu F$$



# Transfer Functions—DC Paths

- Though the bias DC currents $i^-$ and $i^+$ are normally negligible, they are present and should not be ignored.
- A valid solution should ensure that the DC currents $i^-$ and $i^+$ can flow.
  - Otherwise, the operational amplifier will malfunction.
- Note that capacitors block DC current; they should not block $i^-$ and $i^+$.
- For example, the two circuits below would be expected to have the same transfer function. The right circuit, however, is incorrect because it blocks $i^-$.

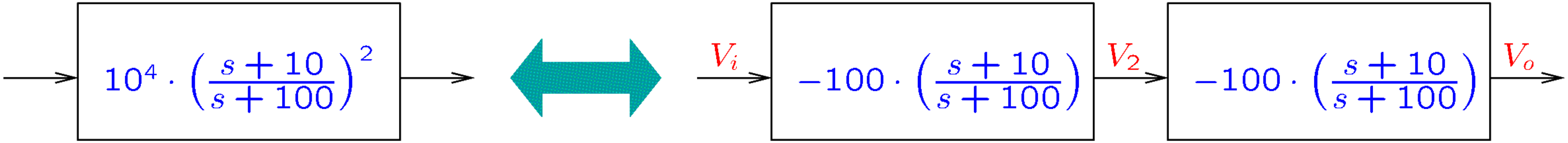


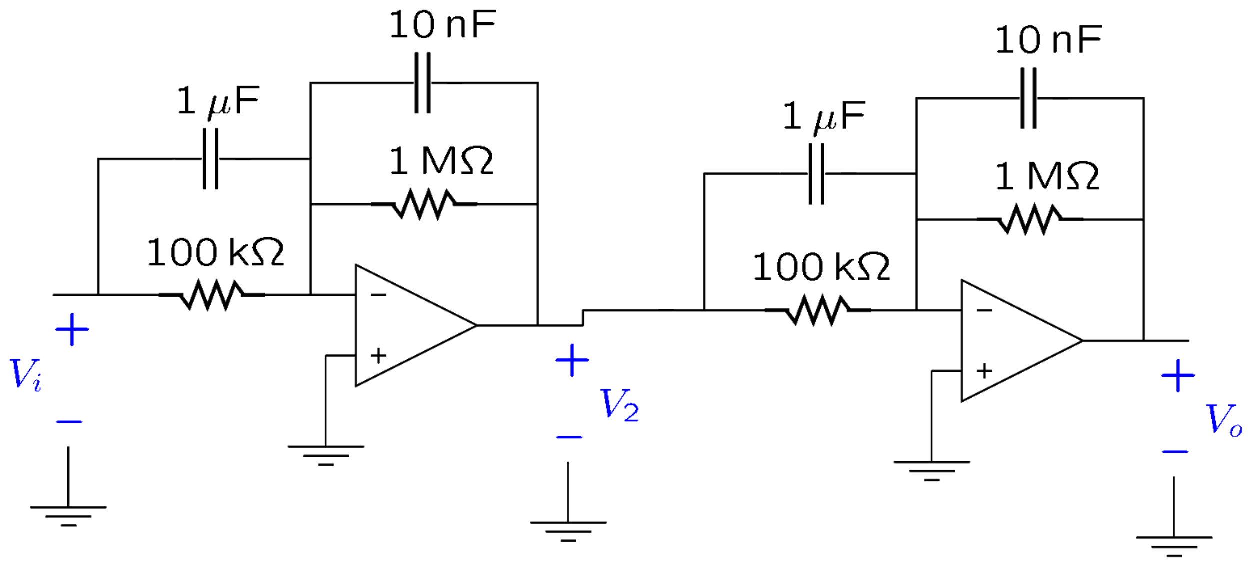



# Transfer Functions—Example 3

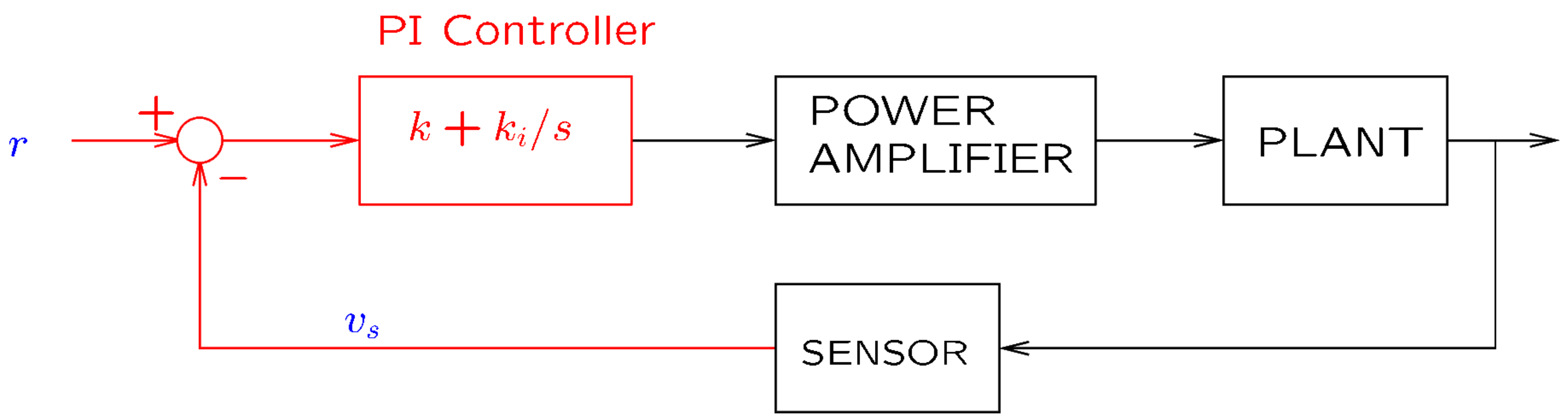
*Implement the transfer function* $10^4 \cdot \left(\frac{s+10}{s+100}\right)^2$.

- *The desired transfer function is the square of the transfer function of Example 2.*
- *Therefore, it can be implemented by two identical cascaded amplifier stages.*



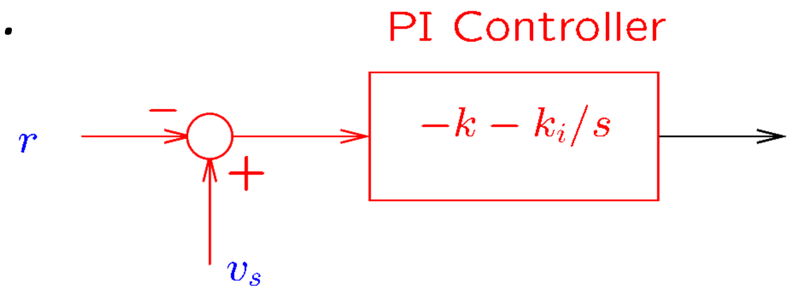



# Transfer Functions—Example 4



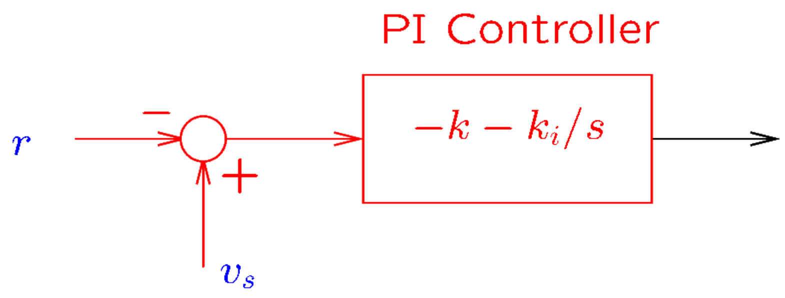

*Implement in hardware the highlighted part of the control system.*

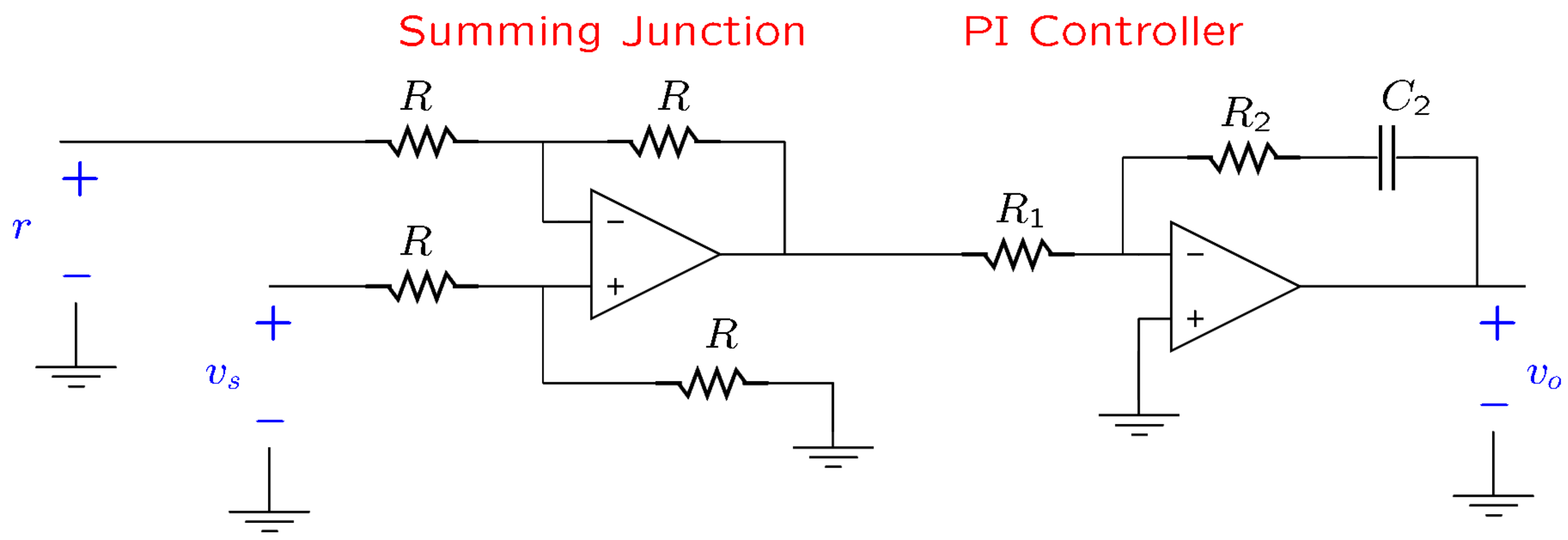
- *Note that transfer functions implemented with the inverting amplifier have negative coefficients.*
- *Therefore, the controller coefficients will be $-k$ and $-k_i$.*
- *To compensate for the minus signs, the signs of the summing junction will be reversed.*



# Transfer Functions—Example 4

- *A possible solution would utilize two cascaded stages:*
  - *The first stage implements the summing junction.*
  - *The second stage implements the PI controller.*

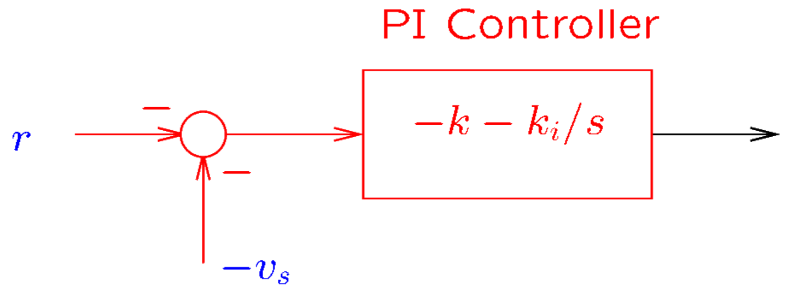


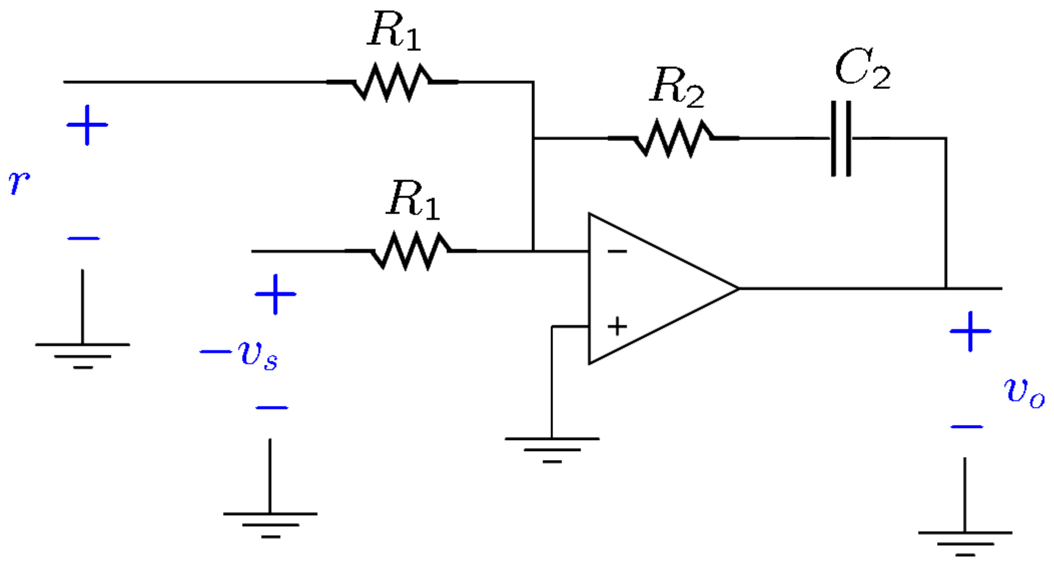




- *However, if the sign of the sensor signal $v_s$ can be changed without an additional stage, there is a simpler solution that uses a single operational amplifier.*

# Transfer Functions—Example 4

- *A simpler solution is to combine the summing junction and the controller into a single stage.*




$$k = \frac{R_2}{R_1} \text{ and } k_i = \frac{1}{R_1 C_2}$$