State-Space Models

Definition. Obtaining State-Space Models.

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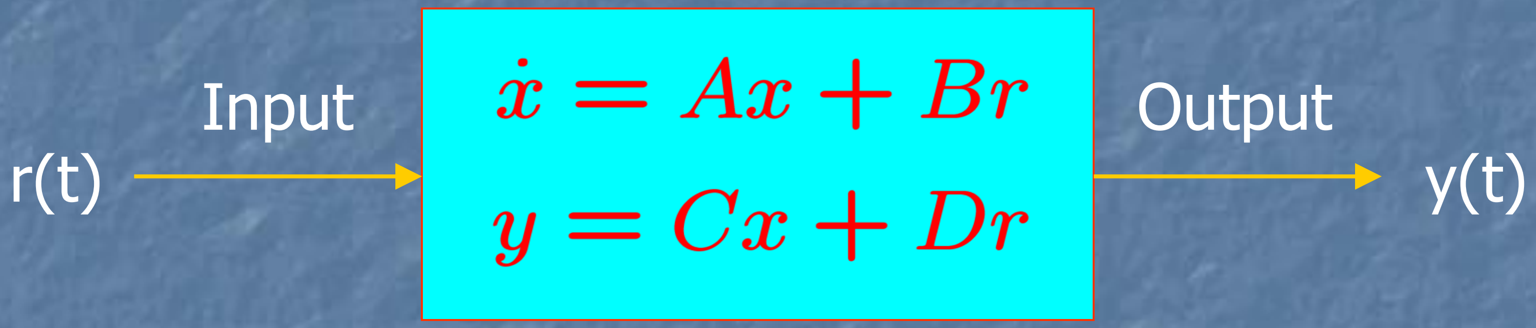
The State Space Representation

- A very common representation is



- A, B, C, D are matrices.
- r, x, and y are vectors.
- x is the state vector.

The State Space Representation

- x: state vector.
  - x may represent system variables (displacement, velocity, voltage, temperature, pressure, etc.)
  - Sometimes x does not have an obvious physical meaning.
- y: output. May represent:
  - Sensor output (frequency, flow rate, current, etc.)
  - Simulation output (variables we wish to visualize).
- r: input (such as force, torque, voltage, etc.)

- SS models: not unique! Equivalent models obtained depending on the choice of x.

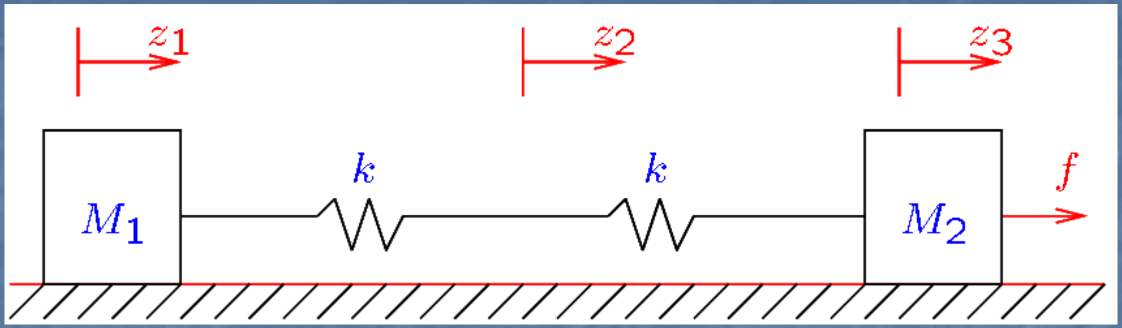
Obtaining state equations

- Write the equations of the system.
- Eliminate redundant variables.
  - Eliminate variables that are a combination of other variables.

    Example: if $v_3 = v_1 + v_2$, substitute v_3 with $v_1 + v_2$.
  - Eliminate variables that appear always with the same order of the derivative.
- Define state variables, inputs, and outputs.
- To obtain the state equations:
  - Solve for $\dot{x}_1, \ldots, \dot{x}_n$ in terms of $x_1, \ldots, x_n$ and $r_1, \ldots, r_m$.
  - Solve for $y_1, \ldots, y_p$ in terms of $x_1, \ldots, x_n$ and $r_1, \ldots, r_m$.

Obtaining state equations

Eliminating redundant variables—an example:



$$
\begin{aligned}
f &= M_2\ddot{z}_3 + k(z_3 - z_2)\\
0 &= k(z_2 - z_3) + k(z_2 - z_1)\\
0 &= M_1\ddot{z}_1 + k(z_1 - z_2)
\end{aligned}
$$

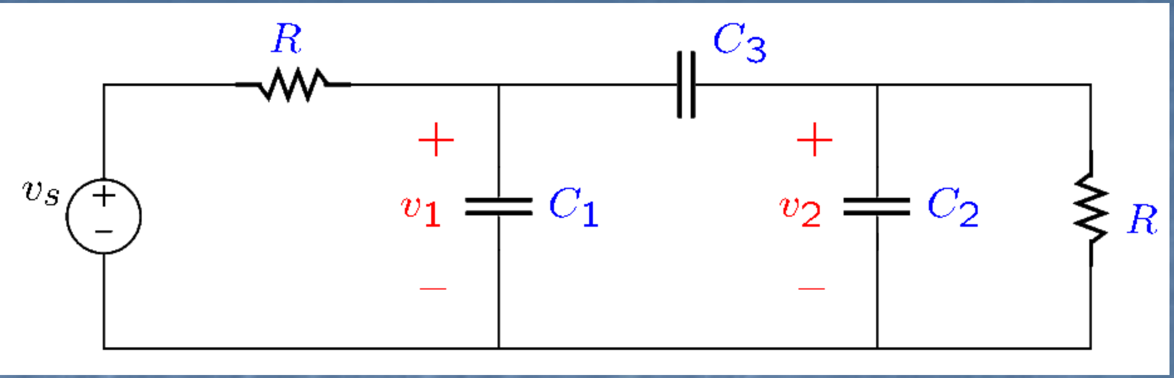
- The variable z_2 should be eliminated.
- After eliminating z_2 we obtain:

$$
\begin{aligned}
f &= M_2\ddot{z}_3 + \frac{k}{2}(z_3 - z_1)\\
0 &= M_1\ddot{z}_1 + \frac{k}{2}(z_1 - z_3)
\end{aligned}
$$

Obtaining state equations

Example for

- Defining state variables.
- Solving for $\dot{x}_1, \dot{x}_2$.



- Let $r = v_s, x_1 = v_1,$ and $x_2 = v_2$.

$$0 = \frac{r - x_1}{R} - C_1\dot{x}_1 + C_3(\dot{x}_2 - \dot{x}_1)$$

$$0 = \frac{x_2}{R} + C_2\dot{x}_2 + C_3(\dot{x}_2 - \dot{x}_1)$$

- When solving for $\dot{x}_1$ and $\dot{x}_2$ we obtain:

$$\dot{x}_1 = \frac{r - x_1 - \frac{x_2 C_3}{C_2 + C_3}}{R(C_1 + C_2||C_3)} \quad \text{and} \quad \dot{x}_2 = \frac{\frac{(r - x_1)C_3}{C_1 + C_3} - x_2}{R(C_2 + C_1||C_3)}$$

Obtaining state equations

- How to choose state variables:
  - For mechanical systems: velocities and displacements; let z be a displacement.
  - For electrical systems:
    - Inductor currents and capacitor voltages.
    - Sometimes, voltages and integrals of voltages; let z be a voltage integral.
- If n > 1 is the highest derivative order of z:
  - Define as state variables

    $$x_n = z^{(n-1)} \quad x_{n-1} = z^{(n-2)} \quad \ldots \quad x_1 = z.$$

  - Include the equations

    $$\dot{x}_{n-1} = x_n \quad \dot{x}_{n-2} = x_{n-1} \quad \ldots \quad \dot{x}_1 = x_2.$$

Obtaining state equations

- Additive constants can be included in variables

  Example: If $Mg = M\ddot{z} + B\dot{z} + kz$, let
  $$x_1 = z - \frac{Mg}{k} \text{ and } x_2 = \dot{x}_1$$
  The equation becomes $0 = M\dot{x}_2 + Bx_2 + kx_1$

- Alternatively, additive constants can be taken as the effect of a constant input.

  Example: If $Mg = M\ddot{z} + B\dot{z} + kz$, let
  $$x_1 = z, x_2 = \dot{x}_1, \text{and } r = Mg$$

Obtaining state equations

- The Derivative Grouping method may be used when the equations contain derivatives of input variables.

Example:

$$\dddot{v} + 2\ddot{v} + 3\ddot{w} + 4\dot{w} + 5v + 6w = 7\dot{r}_1 + 8\ddot{r}_2 + 9r_2$$

Rewrite the equation as:

$$\dddot{v} + 2\ddot{v} + 3\ddot{w} - 8\ddot{r}_2 + 4\dot{w} - 7\dot{r}_1 + 5v + 6w - 9r_2 = 0$$

Let $x_1 = v$, $x_2 = w$, and $\dot{x}_3 = 5v + 6w - 9r_2$.

Substituting, we obtain:

$$\dddot{x}_1 + 2\ddot{x}_1 + 3\ddot{x}_2 - 8\ddot{r}_2 + 4\dot{x}_2 - 7\dot{r}_1 + \dot{x}_3 = 0$$

Let $\dot{x}_4 = 4x_2 - 7r_1 + x_3$. Substituting, we obtain:

$$\dddot{x}_1 + 2\ddot{x}_1 + 3\ddot{x}_2 - 8\ddot{r}_2 + \ddot{x}_4 = 0$$

We will take $\dot{x}_1 = -2x_1 - 3x_2 + 8r_2 - x_4$.

Obtaining state equations

- If the highest order derivative of a variable equals the highest order derivative of an input, the variable could be represented by an output.

Example:

$\ddot{v} + 2\dot{w} + 5v + 4w = 4\ddot{r}_1 + 8r_2 + 3\dot{r}_1$

Let $x_1 = v - 4r_1$, $y_1 = v = x_1 + 4r_1$, and $x_2 = w$.

$\ddot{x}_1 + 4\ddot{r}_1 + 2\dot{x}_2 + 5x_1 + 20r_1 + 4x_2 = 4\ddot{r}_1 + 8r_2 + 3\dot{r}_1$

Let's write the equation in the form

$\ddot{x}_1 + 2\dot{x}_2 - 3\dot{r}_1 + 5x_1 + 4x_2 + 20r_1 - 8r_2 = 0$

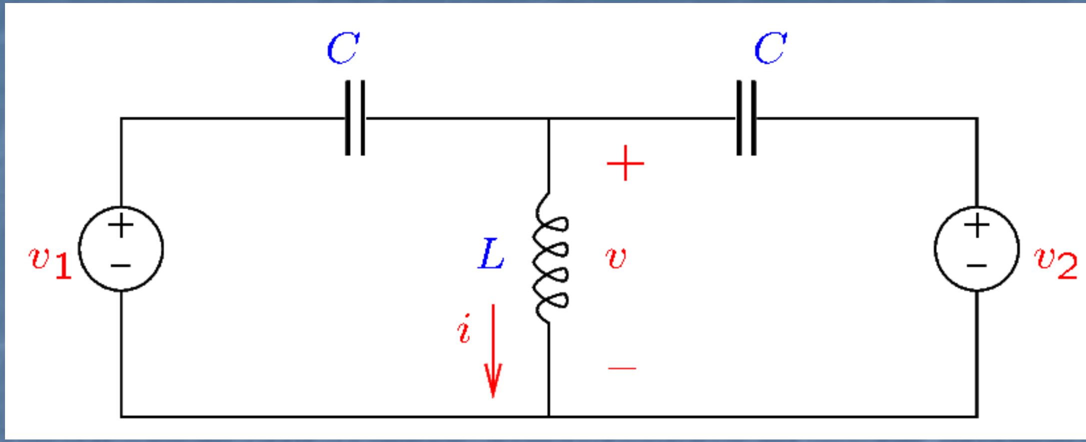
Let $x_3 = 5x_1 + 4x_2 + 20r_1 - 8r_2$. Substituting:

$\ddot{x}_1 + 2\dot{x}_2 - 3\dot{r}_1 + \dot{x}_3 = 0$

We will take $\dot{x}_1 = -2x_2 + 3r_1 - x_3$.

Obtaining state equations

Example: Find a state space model adequate for simulating i and v. The inputs are v_1 and v_2.



Note that $C(\dot{v}_1 - \dot{v}) + C(\dot{v}_2 - \dot{v}) = i$ and $L\frac{di}{dt} = v$.

Let $r_1 = v_1$, $r_2 = v_2$, $x_1 = i$, $x_2 = v - \frac{r_1+r_2}{2}$,

$y_1 = v = x_2 + \frac{r_1+r_2}{2}$, and $y_2 = i = x_1$. We obtain

$\dot{x}_1 = \frac{1}{L}x_2 + \frac{r_1+r_2}{2L}$ and $\dot{x}_2 = -\frac{1}{2C}x_1$.

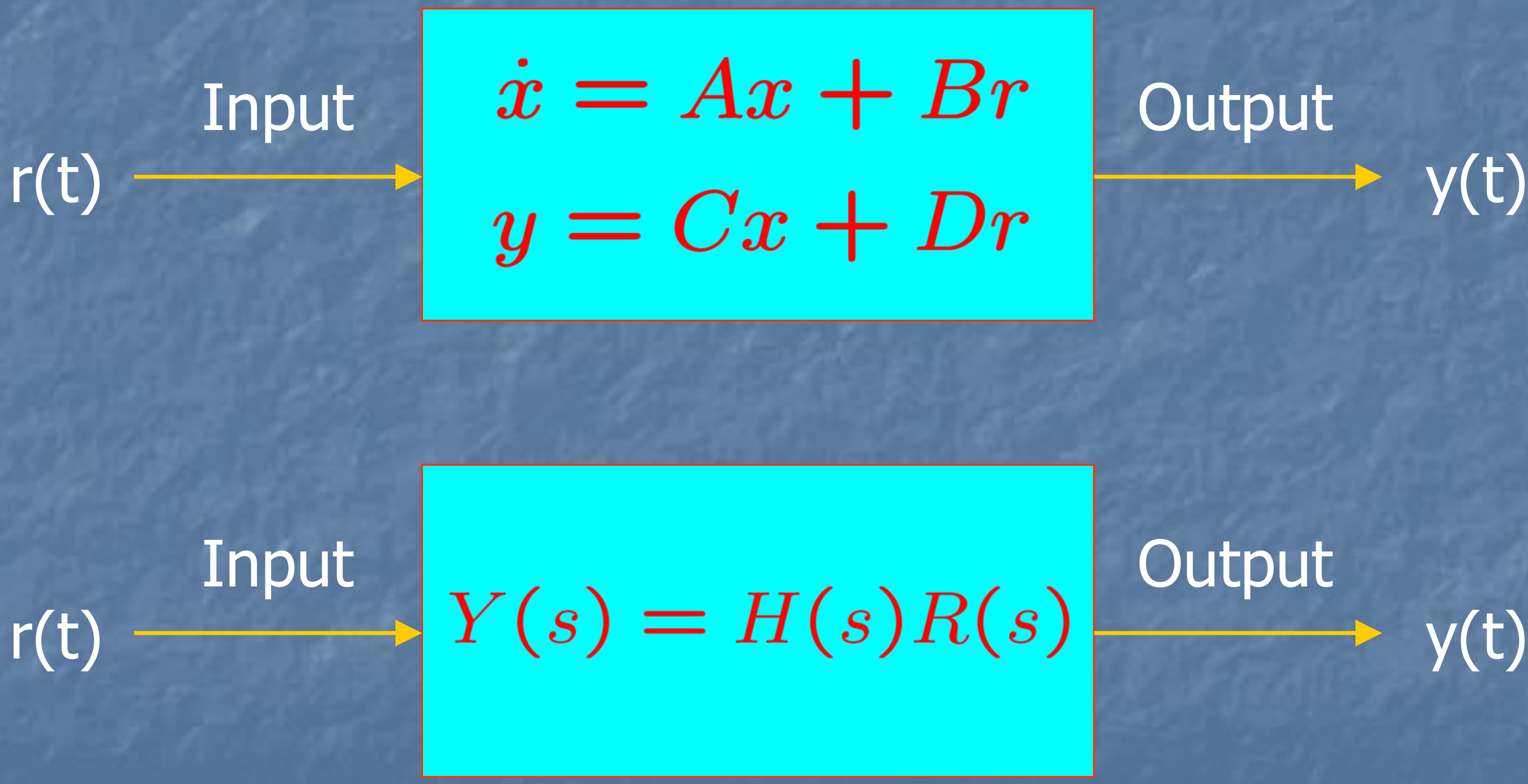
Solution of the SS Equation

- Note that x(t) depends on
  - x(0) – the initial state
  - r(t) – the input
- Let $\phi(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$

$$
\begin{aligned}
X(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}BR(s) \\
x(t) &= \phi(t)x(0) + \int_0^t \phi(t-\tau)Br(\tau)d\tau
\end{aligned}
$$

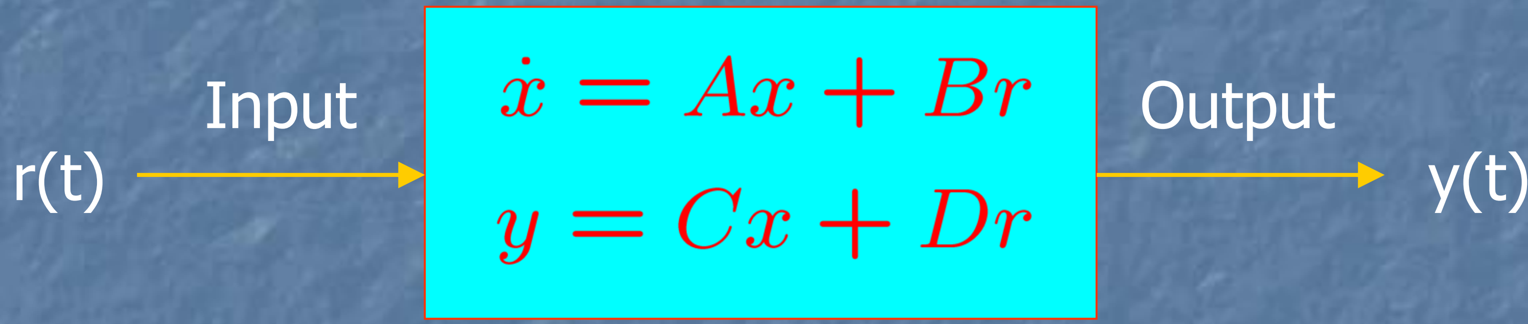
The Relation of SS and TF Models

- Conversions TF → SS and SS → TF possible.



Transfer Function of SS Model

- Assumes zero initial conditions: x(0) = 0.



$$Y(s) = \left(C(sI - A)^{-1}B + D\right) R(s) + C(sI - A)^{-1}x(0)$$

$$\boxed{H(s) = C(sI - A)^{-1}B + D}$$