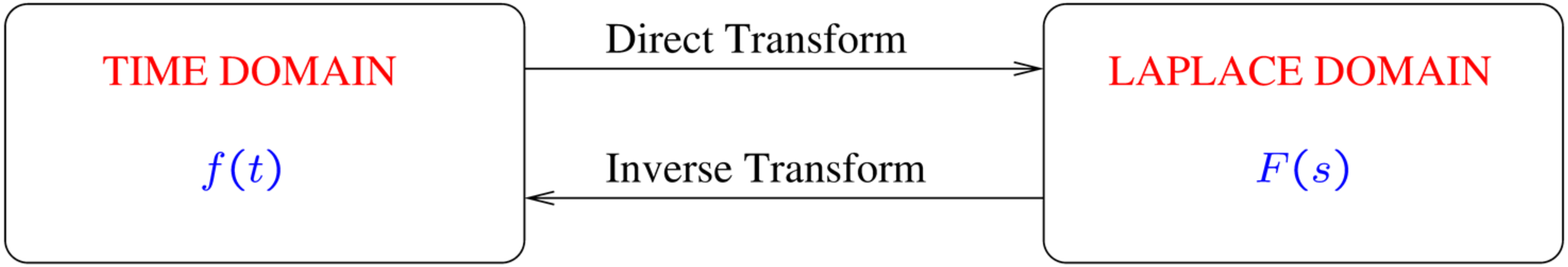
Review of the Laplace Transform

Applications to Differential Equations and System Modeling

M.V. Iordache, *EEGR3523 Mechatronics*, Spring 2019, LeTourneau University

The Laplace Transform

- System models involve differential equations.
- The Laplace transform provides a convenient method to solve differential equations.



- Notation:
  - Lower case for time-domain functions: f(t), g(t), …
  - Upper case for their Laplace transform: F(s), G(s), …

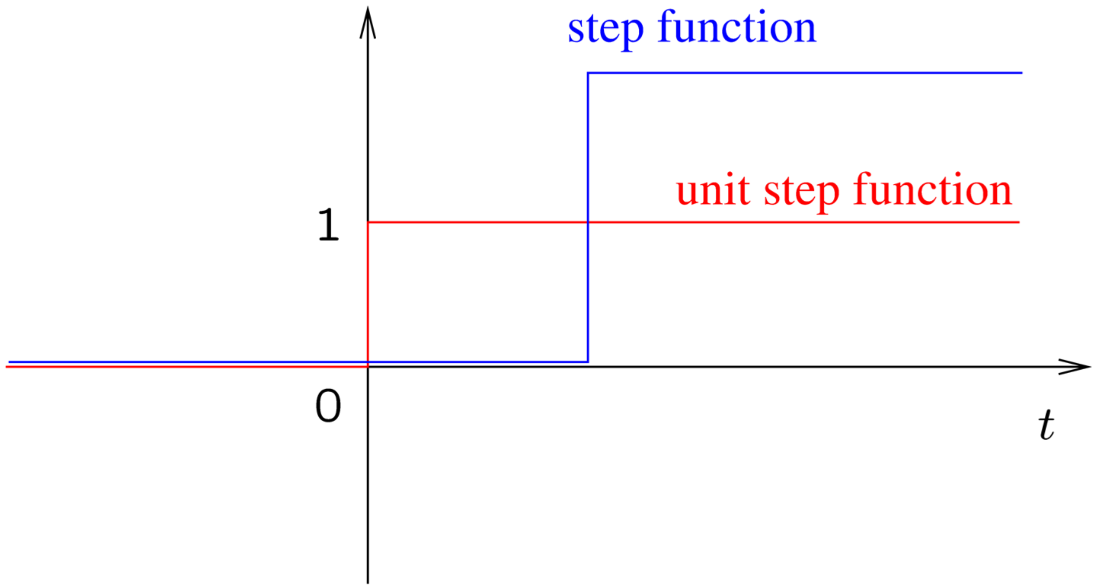
The Laplace Transform

- The Laplace transform converts a function f(t) from the *time domain* to a function F(s) in the *Laplace domain*.

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

- Values of $f(t)$ for t < 0 do not affect $F(s)$!
- The Laplace domain is also known as the *frequency domain*.

The Step Function



- The unit step function is a step function defined as

$$\mathcal{U}(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

The Laplace Transform

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

- Values of $f(t)$ for t < 0 do not affect $F(s)$!
- In the context of the Laplace transform, functions are normally specified for $t \geq 0$.

Example: $f(t) = \sin(3t)$ means $f(t) = \sin(3t)$ for $t \geq 0$;
$f(t)$ may have any values for $t < 0$.

Example: $f(t) = 5$ means $f(t) = 5$ for $t \geq 0$;

Example: If $U(t)$ is the unit step function, then
$5U(t)$ is the same as 5 (since we are in the $t \geq 0$ context).

Applying the Laplace Transform

- Assume a system of differential equations is given.

$$\text{Example: } \dot{z} + 2z = y,\ \dot{y} + y = 4,\ z(0^-) = 0,\ y(0^-) = 3$$

- The Laplace transform can be applied to find the unknowns beginning with time t = 0.
  - Convert the equations to the Laplace domain.

$$sZ + 2Z = Y,\ sY - 3 + Y = \frac{4}{s}$$

  - Solve for the unknowns in the Laplace domain.

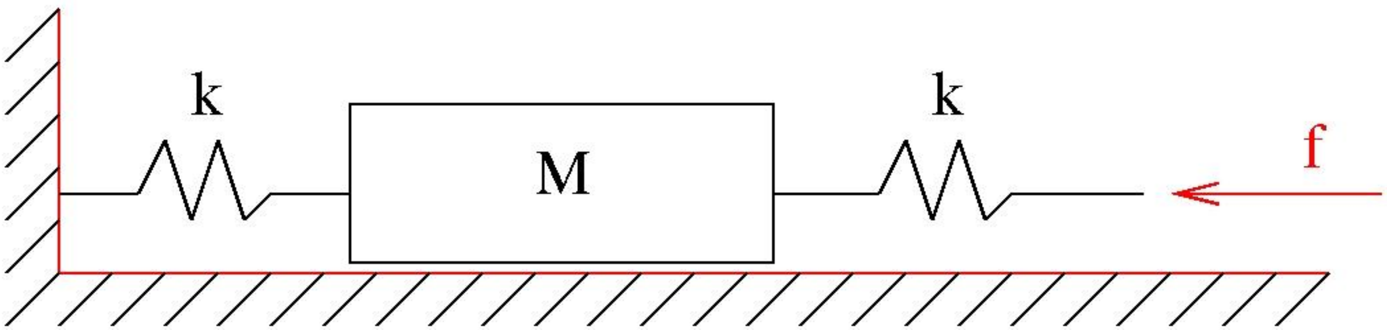
$$Z = \frac{3s+4}{s(s+1)(s+2)},\ Y = \frac{3s+4}{s(s+1)}$$

  - Apply the inverse Laplace transform to convert the result to the time domain.

$$z = 2 - e^{-t} - e^{-2t},\ y = 4 - e^{-t} \text{ for all } t \geq 0$$

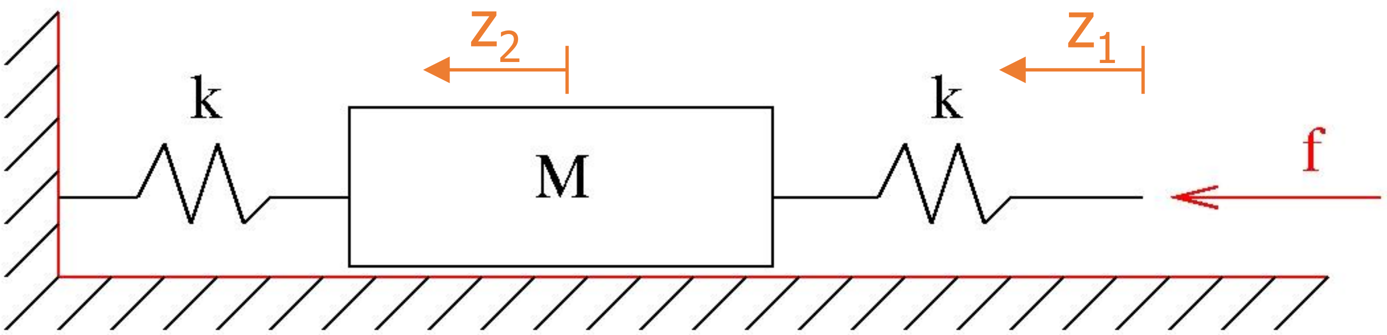
Example

- All components are ideal:
  - No friction
  - Springs of zero mass
- Find the displacements as a function of time.



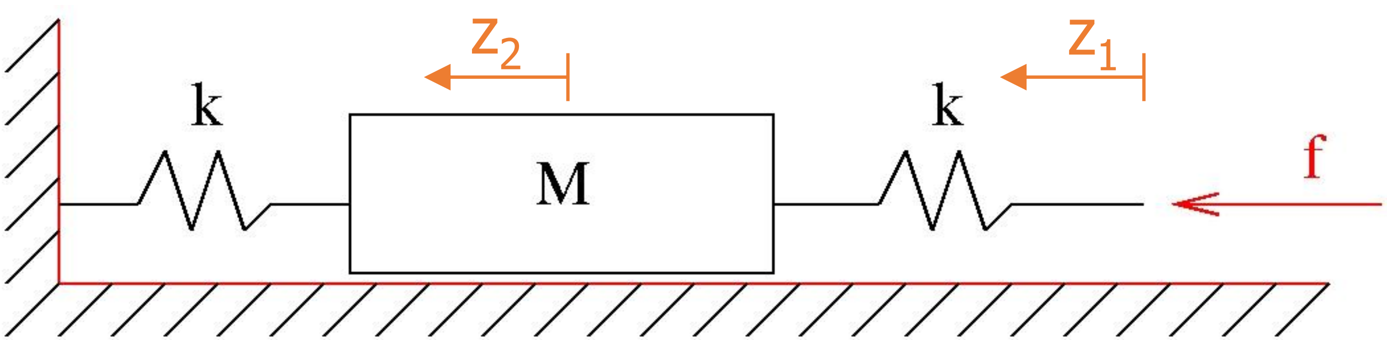
Example

- Given:
  - The initial displacement $z_1$and the initial velocity v_1.
  - The force f.
  - M and k.
- Find the displacements as a function of time.



Example

1. Let t = 0 at the initial moment.
2. Write the equations of the system.
3. [Initial condition calculations → $z_2(0^-)$, $v_2(0^-)$.]
4. Convert the equations to the Laplace domain.
5. Solve for $Z_1(s)$ and $Z_2(s)$.
6. Find $z_1(t)$ and $z_2(t)$ by inverse Laplace transform.



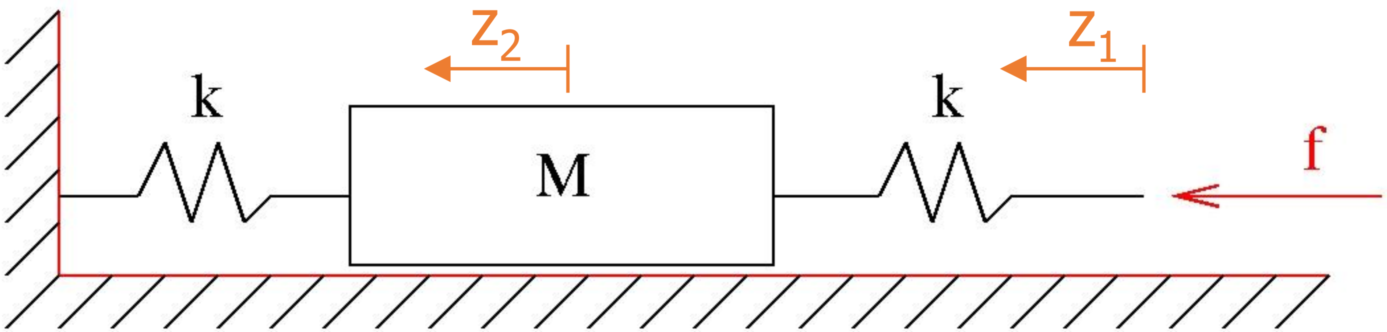
Example—Initial Conditions

- f and $z_1(0^-)$ are given.

$$f = k(z_1 - z_2)$$

$$\Rightarrow f(0^-) = k(z_1(0^-) - z_2(0^-)) \Rightarrow z_2(0^-) = z_1(0^-) - \frac{f(0^-)}{k}$$

$$\Rightarrow \dot{f}(0^-) = k\big(v_1(0^-) - v_2(0^-)\big) \Rightarrow v_2(0^-) = v_1(0^-) - \frac{\dot{f}(0^-)}{k}$$

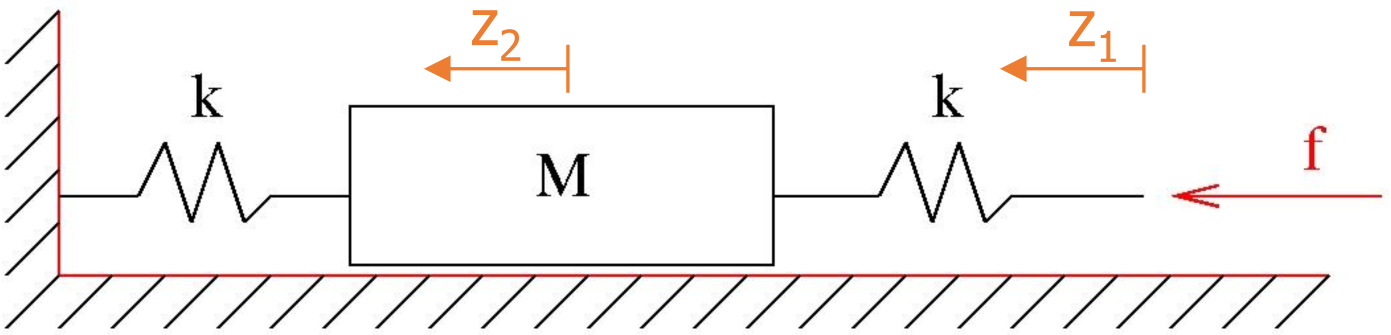


Example—In the Laplace Domain

$$F = k(Z_1 - Z_2)$$

$$0 = Ms^2 Z_2 - Mv_2(0^-) - Msz_2(0^-) + kZ_2 + k(Z_2 - Z_1)$$

$$Z_2 = \frac{F + Mv_2(0^-) + Msz_2(0^-)}{Ms^2 + k} \qquad Z_1 = Z_2 + \frac{F}{k}$$



Example—Back to the Time Domain

Assume $f(t) = f_0 U(t)$. Then, $F = \frac{f_0}{s}$ (see formula handout).

$$Z_2 = \frac{F + Mv_2(0^-) + Msz_2(0^-)}{Ms^2 + k} \qquad Z_1 = Z_2 + \frac{F}{k}$$

$$\Rightarrow z_2(t) = \frac{f_0}{k} - \frac{f_0}{k}\cos\left(\sqrt{\frac{k}{M}}t\right) + v_2(0^-)\sqrt{\frac{M}{k}}\sin\left(\sqrt{\frac{k}{M}}t\right) + z_2(0^-)\cos\left(\sqrt{\frac{k}{M}}t\right)$$

$$\Rightarrow z_1(t) = \frac{2f_0}{k} - \frac{f_0}{k}\cos\left(\sqrt{\frac{k}{M}}t\right) + v_2(0^-)\sqrt{\frac{M}{k}}\sin\left(\sqrt{\frac{k}{M}}t\right) + z_2(0^-)\cos\left(\sqrt{\frac{k}{M}}t\right)$$

Calculating the Laplace Transform

- For direct Laplace transform
  - Apply the formulas on the handout.
- For inverse Laplace transform:
  - Apply partial fraction decomposition first when necessary.
  - Apply the formulas on the handout.

Most Common Mistakes

$\mathcal{L}\{f(t)g(t)\} \neq F(s)G(s)$, but

$$\boxed{\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)}$$

$\mathcal{L}\{1\} \neq 1$, but $\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$

Example:

$$\mathcal{L}\{2 + 3t\} = \mathcal{L}\{2\} + 3\mathcal{L}\{t\} = \frac{2}{s} + \frac{3}{s^2}$$

Alternative Method for Inverse LT (Optional)

- Assume $F(s) = \frac{P(s)}{Q(s)}$, where P and Q are polynomials.
- Since $Q(s)$ is a polynomial, it can be factored as

$$Q(s) = c \cdot (s - p_1)^{n_1} \cdot (s - p_2)^{n_2} \cdot \ldots \cdot (s - p_m)^{n_m}$$

where $p_1, p_2, \ldots, p_m$ are the roots of Q, and n_k is the multiplicity of the root p_k.

- Let R_i denote the *residue* of $F(s)e^{st}$ at the root p_i:

$$R_i = \frac{1}{(n_i - 1)!} \cdot \lim_{s \to p_i} \frac{d^{n_i - 1}}{ds^{n_i - 1}} \left[(s - p_i)^{n_i} F(s) e^{st} \right]$$

- Then:

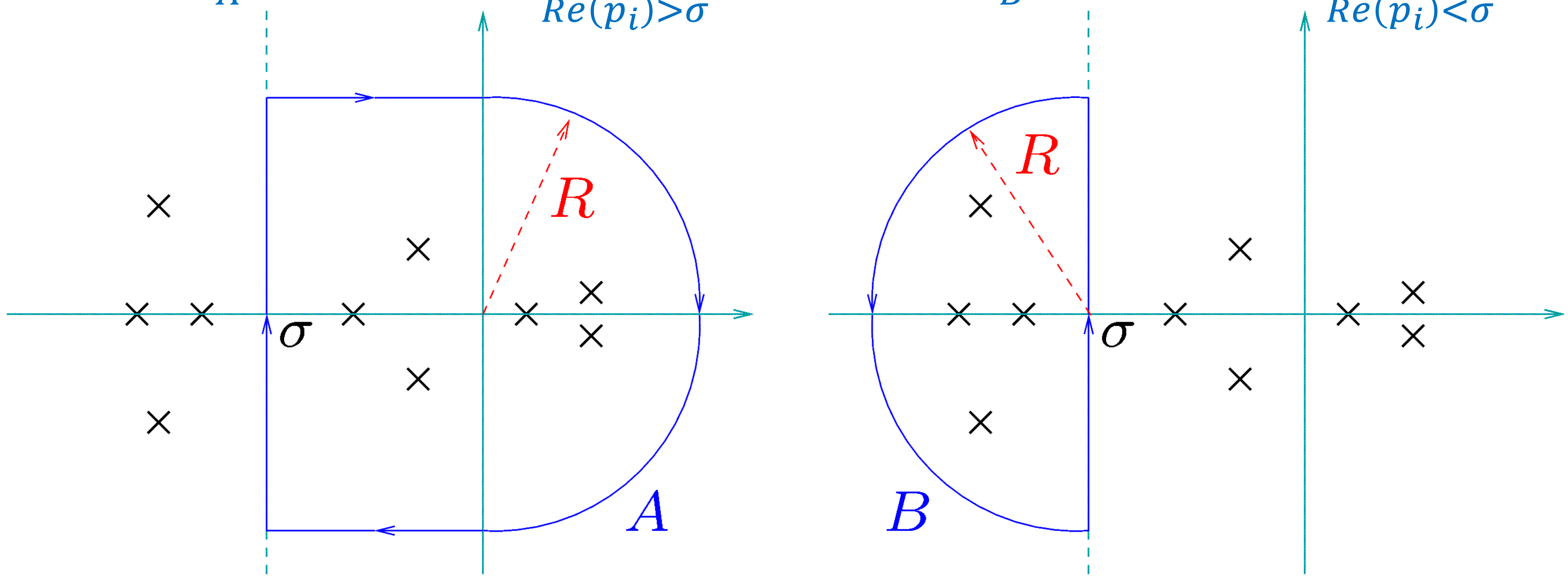
$$f(t) = R_1 + R_2 + \cdots + R_m$$

Alternative Method for Inverse LT—Justification

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds = \frac{1}{2\pi i}\cdot\lim_{R\to\infty}\int_{\sigma-iR}^{\sigma+iR} F(s)e^{st}ds$$

By Cauchy's Theorem:

$$-\frac{1}{2\pi i}\int_A F(s)e^{st}ds = \sum_{Re(p_i)>\sigma} R_i \text{ and } \frac{1}{2\pi i}\int_B F(s)e^{st}ds = \sum_{Re(p_i)<\sigma} R_i$$

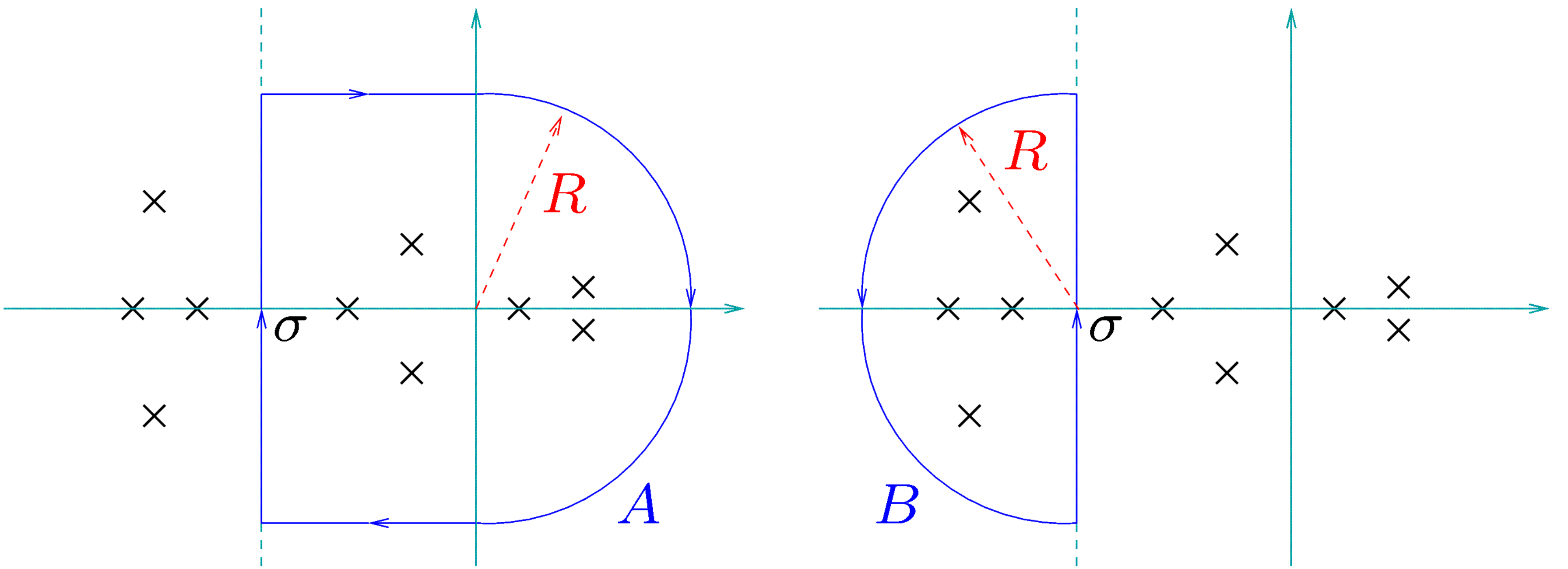


The figure shows the curves A and B in the complex plane; the position of a root p_i is marked by an x.

Alternative Method for Inverse LT—Justification

As $R \to \infty$:

- The integral along the horizontal lines vanishes.
- If $t < 0$, the integral along the semicircle of A vanishes.
- If $t > 0$, the integral along the semicircle of B vanishes.



Alternative Method for Inverse LT—Justification

Therefore:

- For $t < 0$, $f(t) = -\sum_{Re(p_i)>\sigma} R_i$.
- For $t > 0$, $f(t) = \sum_{Re(p_i)<\sigma} R_i$.

In other words:

$$f(t) = \left(\sum_{Re(p_i)<\sigma} R_i\right) U(t) - \left(\sum_{Re(p_i)>\sigma} R_i\right) U(-t)$$

Since we use the unilateral Laplace transform, the Laplace transform is calculated with $Re(s) > Re(p_i)$ for all p_i (or else the integral would not converge). Therefore,

$$f(t) = (R_1 + R_2 + \cdots + R_m)U(t)$$

Alternative Method for Inverse LT—Example

Assume $F(s) = \frac{2s}{(s-1)(s+1)}$.

If $F(s)$ is obtained with the bilateral Laplace transform

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

the following solutions are possible:

1. If $Re(s) < -1$, then $f(t) = -(e^{-t} + e^{t})U(-t)$.
2. If $-1 < Re(s) < 1$, then $f(t) = e^{-t}U(t) - e^{t}U(-t)$.
3. If $Re(s) > 1$, then $f(t) = (e^{-t}+e^{t})U(t)$.

Case 3 is the only possibility with the unilateral Laplace transform

$$F(s) = \int_{-0}^{\infty} f(t)e^{-st}dt$$

This unilateral Laplace transform is used exclusively in our class.