# Stability of Control Systems

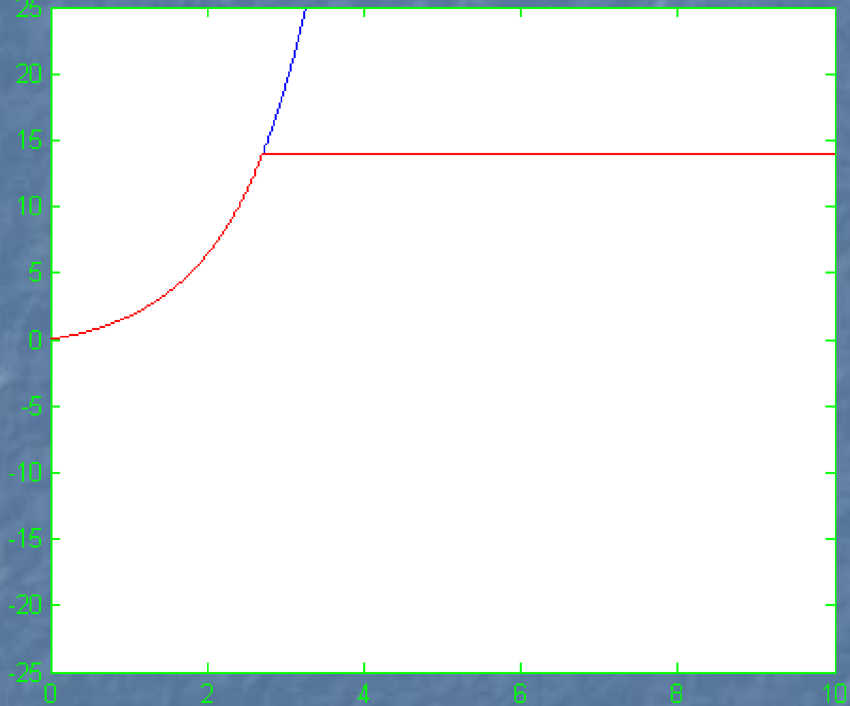
## BIBO Stability. The Routh-Hurwitz Criterion.

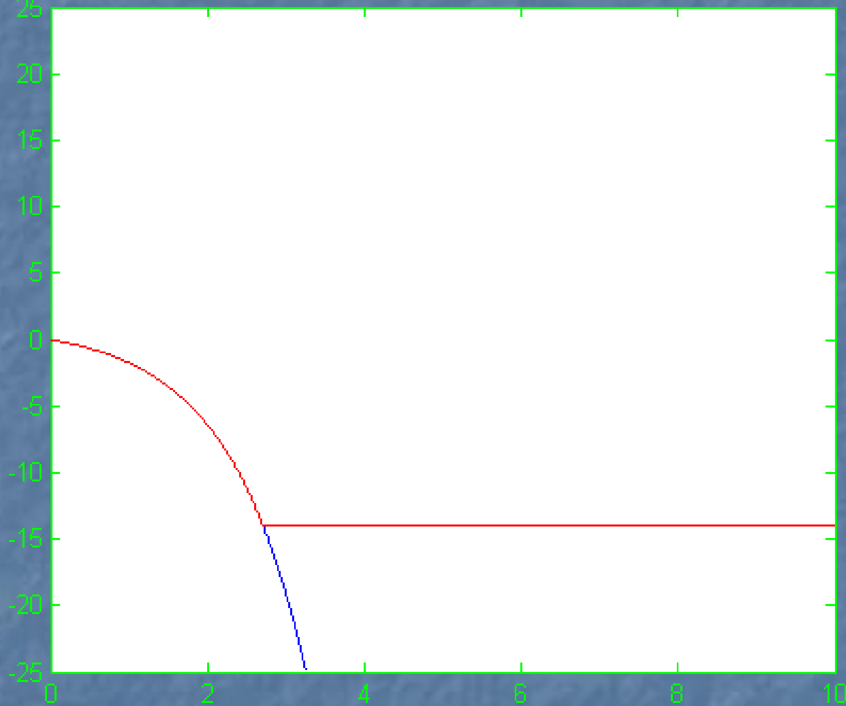
M.V. Iordache, *EEGR3523 Mechatronics*, Spring 2019, LeTourneau University

# Control Systems—Stability

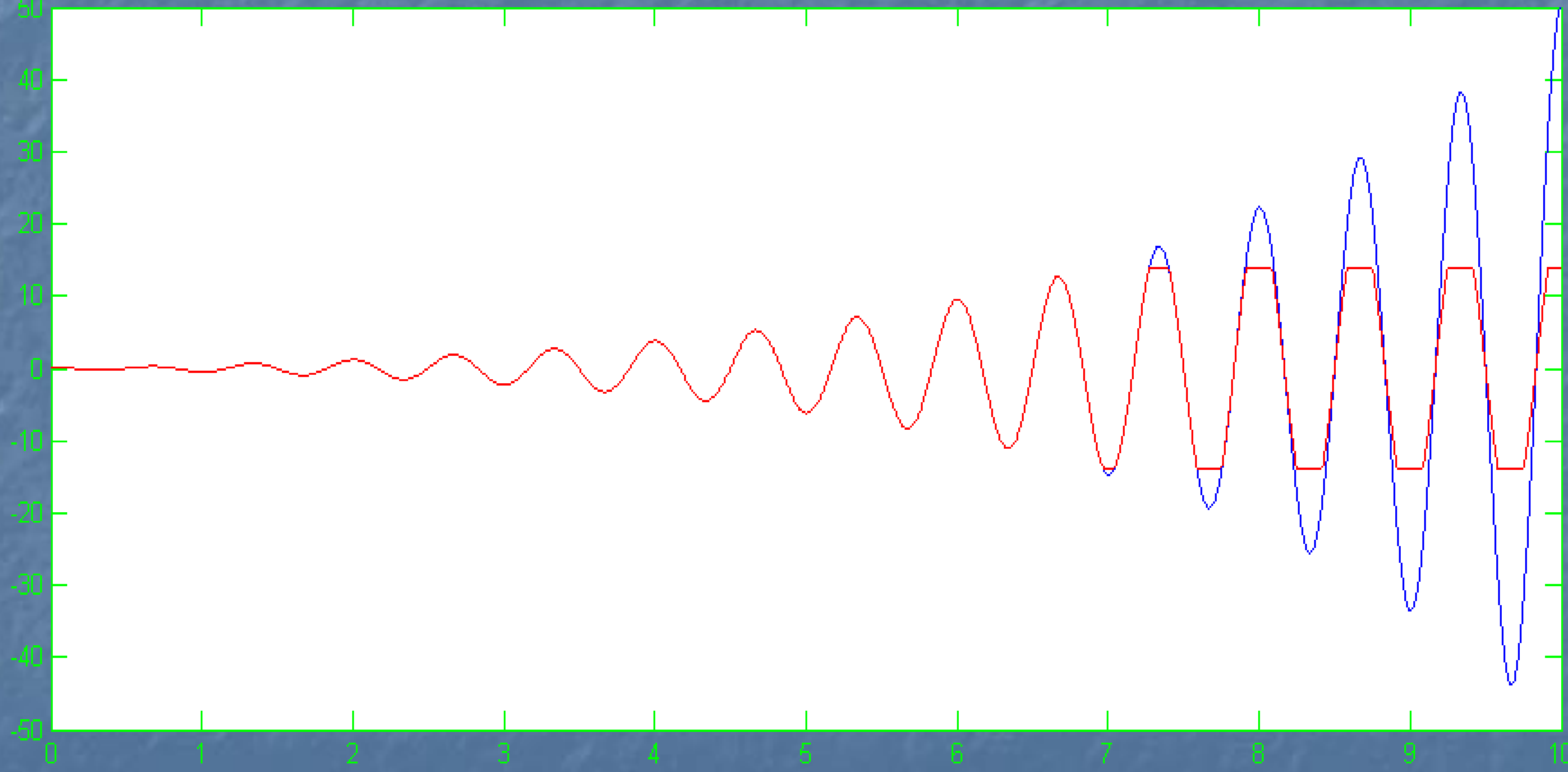
- Stability is essential!
- How instability manifests:
  - The system does not respond to inputs AND
  - The output oscillates OR is saturated at maximum high/low level.

# Manifestations of Instability



# Manifestations of Instability



# Why Study Stability

- Control systems are designed to enhance the performance of a plant. However:
  - An improper selection of control parameters can cause instability even when the plant is stable.
  - A proper selection of control parameters will result in stability even when the plant is unstable.

# BIBO Stability

- A system model that responds to every bounded input with a bounded output is said to be BIBO stable.
- Assuming reasonably accurate models, *the linear model of a stable system will be BIBO stable*.
- A linear system is BIBO stable if and only if it has no poles outside of the LHP.
  - The Routh array and the Routh-Hurwitz criterion can be used to test whether there are poles outside of the LHP.
  - More importantly, they can be used to select control parameters that ensure stability.

# The Routh Array

- Given

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0$$

- define

$$\begin{array}{lllll}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_1 & b_2 & b_3 & \ldots \\
s^{n-3} & c_1 & c_2 & c_3 & \ldots \\
\vdots & \vdots & & & \\
s^0 & h_1 & & &
\end{array}$$

# The Routh Array

| | | | | |
|---|---|---|---|---|
| $s^n$ | $a_n$ | $a_{n-2}$ | $a_{n-4}$ | $\cdots$ |
| $s^{n-1}$ | $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $\cdots$ |
| $s^{n-2}$ | $b_1$ | $b_2$ | $b_3$ | $\cdots$ |
| $s^{n-3}$ | $c_1$ | $c_2$ | $c_3$ | $\cdots$ |
| $\vdots$ | $\vdots$ | | | |
| $s^0$ | $h_1$ | | | |

$$b_1 = a_{n-2} - \frac{a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = a_{n-4} - \frac{a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = a_{n-3} - \frac{a_{n-1} b_2}{b_1}$$

$$c_2 = a_{n-5} - \frac{a_{n-1} b_3}{b_1}$$

# The Routh-Hurwitz Criterion

$$
\begin{array}{lllll}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
\vdots & \vdots & & & \\
s^0 & h_1 & & &
\end{array}
$$

- All roots in LHP if and only if $a_n, a_{n-1}, b_1, c_1, \ldots h_1$ are nonzero and with the same sign.
- If all roots in LHP, then $a_n, a_{n-1}, a_{n-2}, \ldots a_0$ are also nonzero and with the same sign.

# The Routh-Hurwitz Criterion

- Given

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots a_1 s + a_0$$

- Check first whether $a_n, a_{n-1}, a_{n-2}, \ldots a_0$ are nonzero and with the same sign.
- If yes, then find the Routh array and check the first column.
  - The Routh array is not necessary when n < 3.

# The Routh-Hurwitz Criterion

- Used to determine the range of the control parameters for which a control system is stable.
- It can be applied to systems of any order.

# PID Control

- Relates the controller input $e$ to the controller output $u$ with the equation:

$$u = k_p e + k_i \int_0^t e(\tau)d\tau + k_d \frac{de}{dt}$$

- The transfer function is

$$\frac{U(s)}{E(s)} = k_p + \frac{k_i}{s} + k_d s$$

- Special cases:
  - P (proportional) when $k_p \neq 0$
  - PD (proportional + derivative) when $k_p \neq 0$ and $k_d \neq 0$.
  - PI (proportional + integral) when $k_p \neq 0$ and $k_i \neq 0$.