The Root Locus

The root locus is the locus of the closed-loop poles when one gain k is varied. The following rules can be used to sketch or interpret a root locus.

Assume that the equation of the closed-loop poles has the form

$$1 + L(k, s) = 0 \tag{1}$$

1. Determine $F(s)$ such that $1 + kF(s) = 0$. This can be done by substituting first $k = -1/F(s)$ in (1) and then solving for $F(s)$.

2. Note that F can be written in the form
$$F(s) = c \cdot \frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$
where c is a constant. Let $k_c = kc$.

3. If $k_c > 0$, a point x on the real axis belongs to the root locus if the total number of poles and zeros of F to the right of x is odd.

   If $k_c < 0$, a point x on the real axis belongs to the root locus if the total number of poles and zeros of F to the right of x is even. Note that zero is even.

4. Let n be the number of poles of F and m the number of zeros of F. The root locus has $|n - m|$ asymptotes.

   The asymptotes meet the real axis at
$$\sigma = \frac{\sum p_i - \sum z_j}{n - m}$$
   where p_i and z_j are the poles and zeros of F.

   The angles of the asymptotes with respect to the real axis are as follows:
$$\text{If } k_c > 0, \quad \theta_l = \frac{\pi + 2\pi l}{n - m}, \text{ for } l = 0, 1, 2, \ldots n - m - 1.$$
$$\text{If } k_c < 0, \quad \theta_l = \frac{2\pi l}{n - m}, \text{ for } l = 0, 1, 2, \ldots n - m - 1.$$

5. The root locus has $\max(n, m)$ branches. As k is varied from 0 to ∞, n branches start from the n poles of F and m branches end at the m zeros of F.

   If $n > m$, then $n - m$ branches go to ∞. If $n < m$, then $m - n$ branches start from ∞.

   The $|n - m|$ branches ending or starting at infinity converge to the asymptotes as they approach infinity.

6. The root locus is symmetric with respect to the real axis.

7. If two branches intersect at s_0, then $\left.\frac{dF}{ds}\right|_{s=s_0} = 0$.

8. Use $|k| = 1/|F(p)|$ to determine the value of k for which a closed-loop pole p is achieved.