The Impedance

Impedance Review. Complex Number Review. Admittance. Application to Voltage and Current Division.

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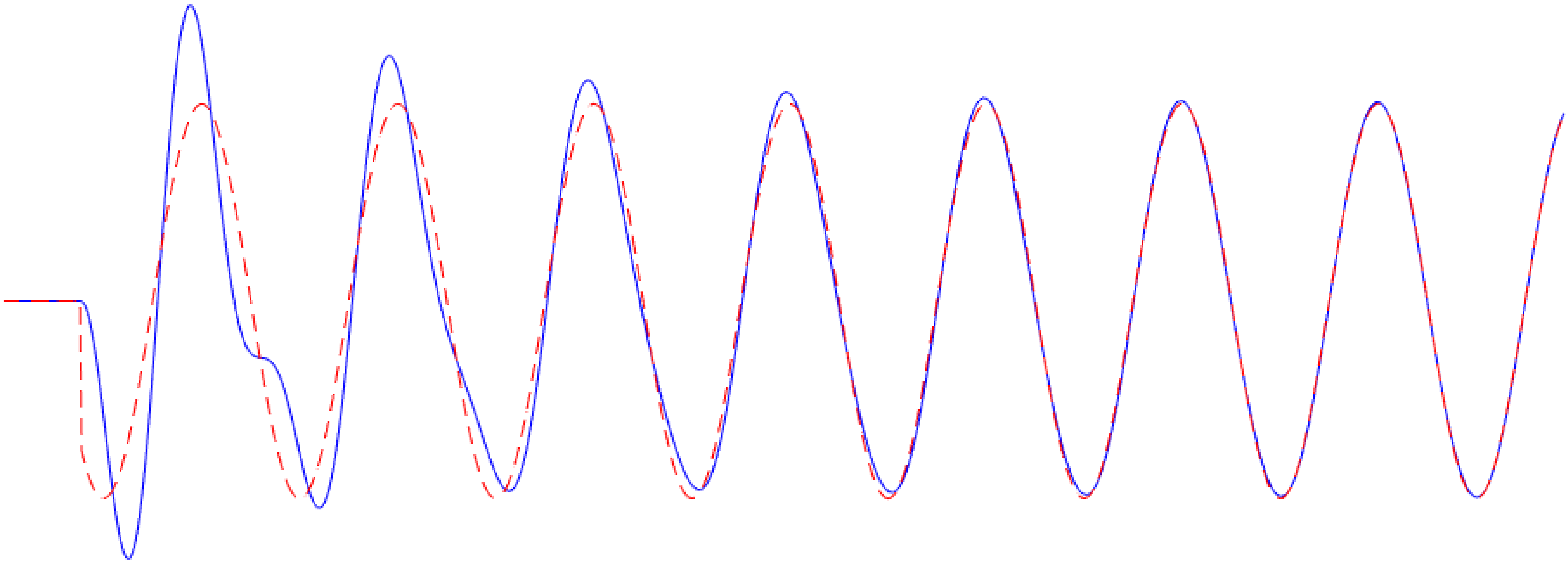
Review

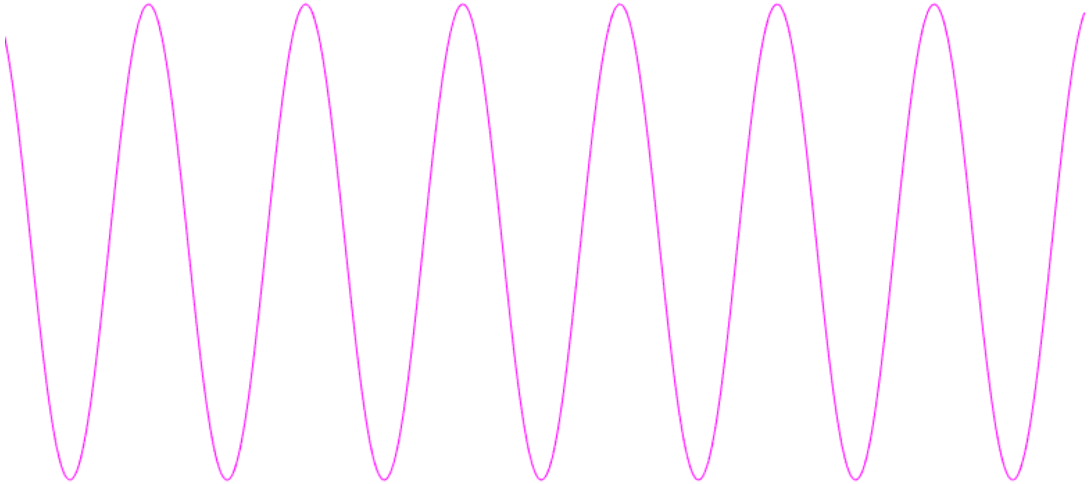
- When an AC circuit is turned on, there are transients (the blue curve).
- *In time, voltages and currents converge to steady-state values (the red curve).*
- Phasor analysis is the fastest way to find steady-state voltages and currents.



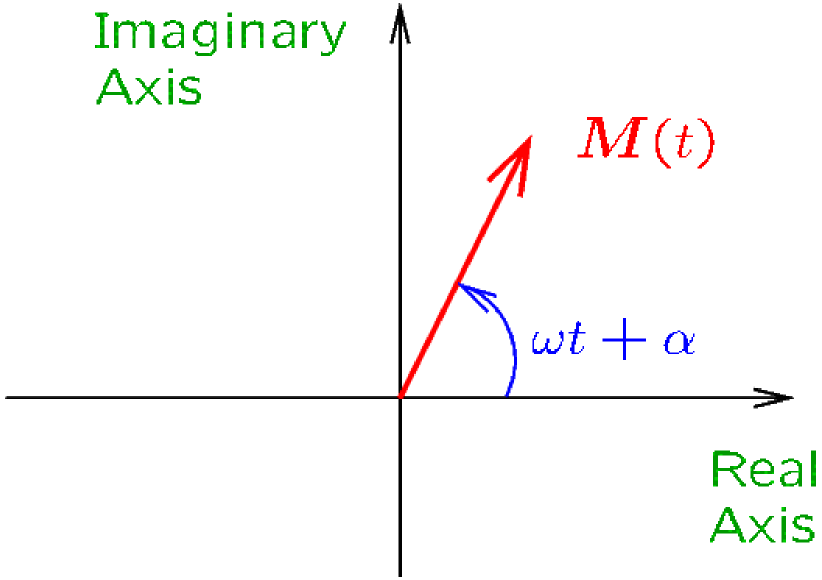
Review—Phasors

- In the time domain, the signal $m(t) = M\cos(\omega t + \alpha)$ is a sinusoidal function.
- In the frequency domain, the signal is a vector $\boldsymbol{M}(t) = M\angle\omega t + \alpha$ that rotates counterclockwise in the complex plane at an angular velocity ω.
- Such rotating vectors are called **PHASORS**.
- We use an *abbreviated phasor notation:* $\boldsymbol{M} = M\angle\alpha$ (the ωt term is dropped).

TIME DOMAIN



FREQUENCY DOMAIN



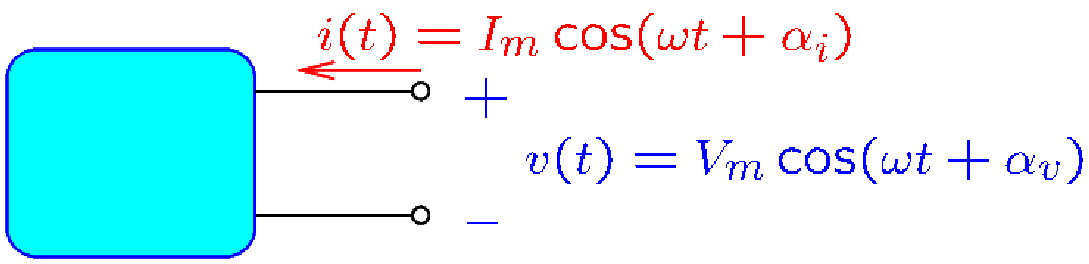
Review—Impedance

- Assume a network with two terminals that has *no independent sources*.
- A voltage $v(t) = V_m \cos(\omega t + \alpha_v)$ will result in a sinusoidal steady-state current of the same frequency: $i(t) = I_m \cos(\omega t + \alpha_i)$.
- In the frequency domain, these correspond to the phasors

$$\boldsymbol{V}(t) = V_m e^{j\omega t + j\alpha_v} \text{ and } \boldsymbol{I}(t) = I_m e^{j\omega t + j\alpha_i}.$$

- For a linear network, the ratio $\boldsymbol{Z} = \boldsymbol{V}(t)/\boldsymbol{I}(t)$ is a constant called **impedance**.

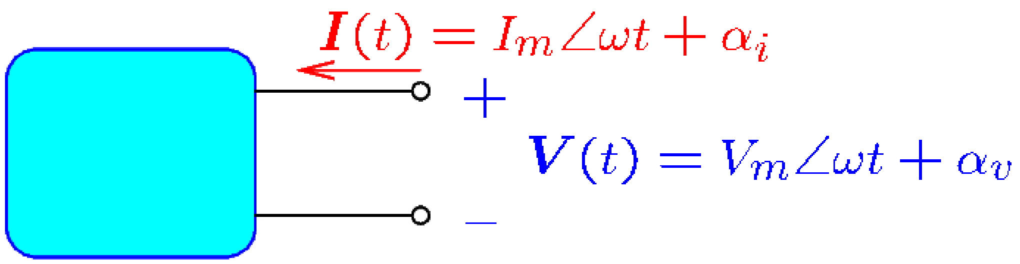
TIME DOMAIN



$$i(t) = I_m \cos(\omega t + \alpha_i)$$

$$v(t) = V_m \cos(\omega t + \alpha_v)$$

FREQUENCY DOMAIN



$$\boldsymbol{I}(t) = I_m \angle \omega t + \alpha_i$$

$$\boldsymbol{V}(t) = V_m \angle \omega t + \alpha_v$$

$$\boldsymbol{Z} = \frac{\boldsymbol{V}(t)}{\boldsymbol{I}(t)} = \frac{V_m}{I_m} \angle \alpha_v - \alpha_i$$

Review—Impedance

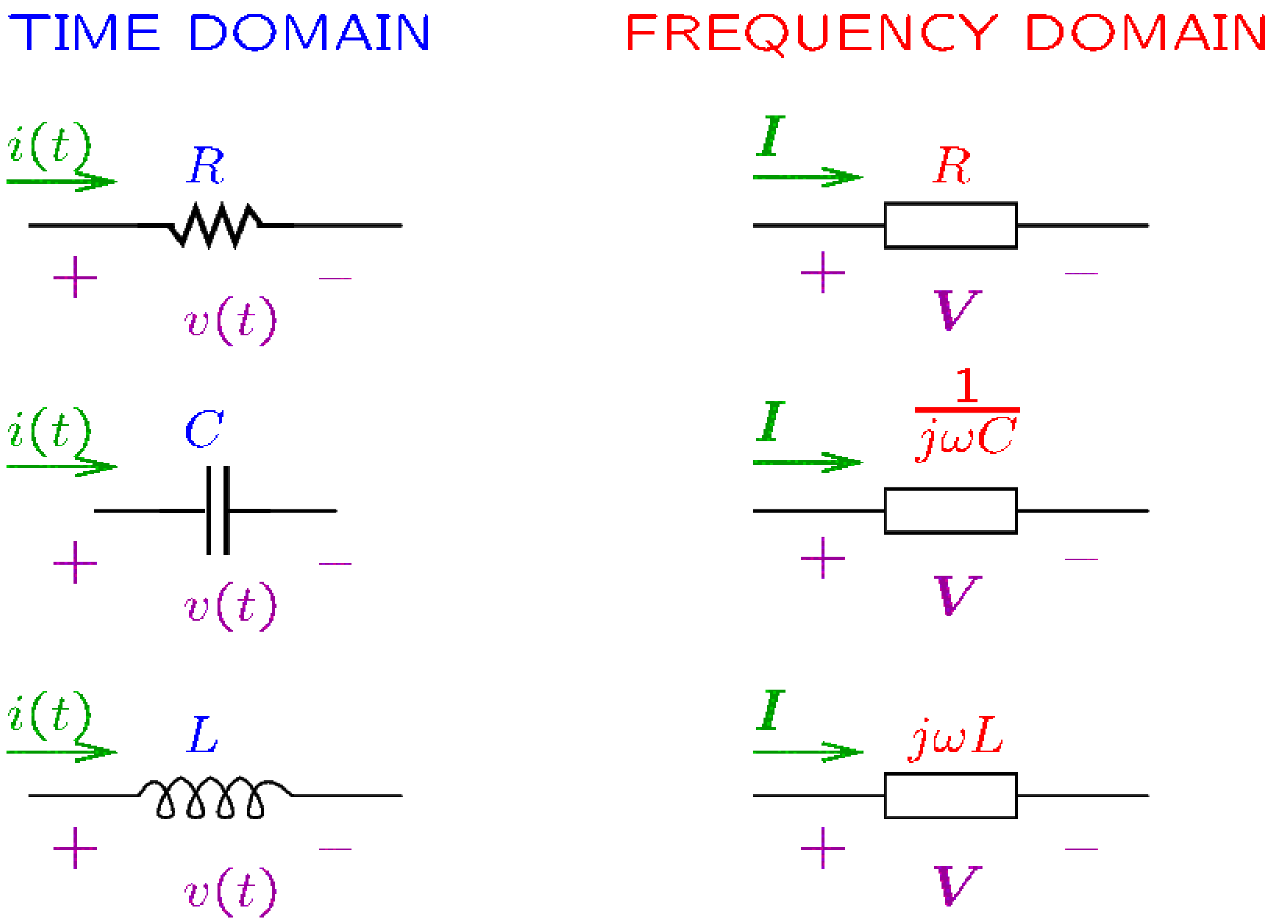
- Since $\boldsymbol{V}(t) = V_m e^{j\omega t + j\alpha_v}$ and $\boldsymbol{I}(t) = I_m e^{j\omega t + j\alpha_i}$, impedance is independent of time:

$$\boldsymbol{Z} = \frac{\boldsymbol{V}(t)}{\boldsymbol{I}(t)} = \frac{V_m}{I_m} e^{j(\alpha_v - \alpha_i)}$$

- *The impedance is a constant, not a phasor!*
- *Only voltages and currents are phasors.*
- The impedance unit is Ohms [Ω].
- *Impedance resembles resistance; it could be viewed as an "AC resistance".*

Review—Impedance

Resistors, capacitors, and inductors are represented in the frequency domain by their impedance. *Memorize the impedance formulas!*

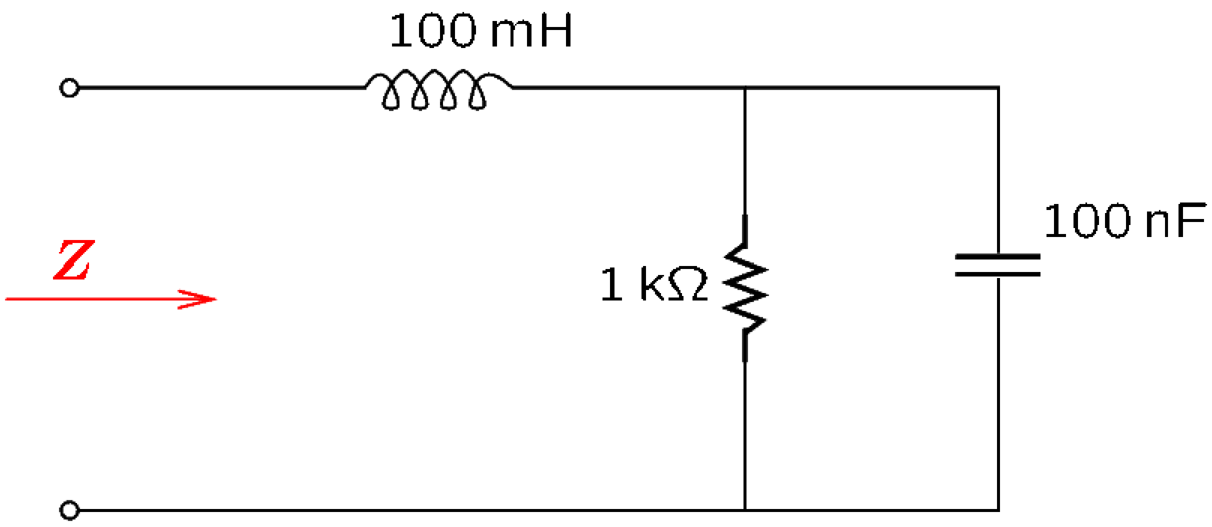


Impedance Combinations

- The methods learned for DC circuits apply also to AC circuits *in the frequency domain*.
- Series and parallel impedances are combined just like resistors.

Example: Find the impedance of the shown network. Assume $f = \frac{10}{2\pi} kHz$.

- The angular frequency is: $\omega = 2\pi f$.
- The resistor impedance: $\boldsymbol{Z_R} = 1\ k\Omega$.
- The inductor has $\boldsymbol{Z_L} = j\omega L = j\ k\Omega$.
- The capacitor has $\boldsymbol{Z_C} = \frac{1}{j\omega C} = \frac{1}{j}\ k\Omega$.
- Overall, $\boldsymbol{Z} = \boldsymbol{Z_L} + (\boldsymbol{Z_R} || \boldsymbol{Z_C})$.

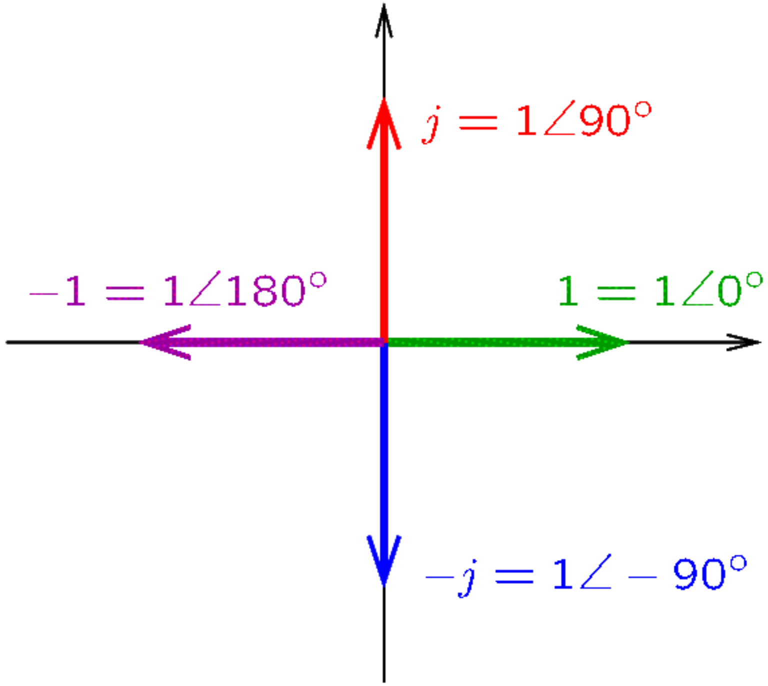


Complex Numbers—Review

To finish our example, let us review first some complex number concepts.

$$(m\angle\alpha)\cdot(n\angle\beta) = me^{j\alpha}ne^{j\beta} = mne^{j\alpha+j\beta} = mn\angle\alpha+\beta$$

$$\frac{m\angle\alpha}{n\angle\beta} = \frac{me^{j\alpha}}{ne^{j\beta}} = \frac{m}{n}e^{j\alpha-j\beta} = \frac{m}{n}\angle\alpha-\beta$$



In our example:

$$\boldsymbol{Z_C} = \frac{1}{j} = \frac{1\angle 0^\circ}{1\angle 90^\circ} = 1\angle -90^\circ = -j\ k\Omega$$

Complex Numbers—Review

$$\boldsymbol{Z_c}||\boldsymbol{Z_R} = \frac{1}{\frac{1}{\boldsymbol{Z_C}} + \frac{1}{\boldsymbol{Z_R}}} = \frac{1}{\frac{j}{10^3} + \frac{1}{10^3}} = \frac{10^3}{1+j}\ \Omega = \frac{1}{1+j}\ k\Omega$$

There are two ways to simplify this expression.

Method 1: *Convert* $1 + j$ *to polar form*

$$\frac{1}{1+j} = \frac{1}{\sqrt{1^2+1^2}\angle \tan^{-1}\frac{1}{1}} = \frac{1\angle 0^\circ}{\sqrt{2}\angle 45^\circ} = \frac{1}{\sqrt{2}}\angle -45^\circ$$

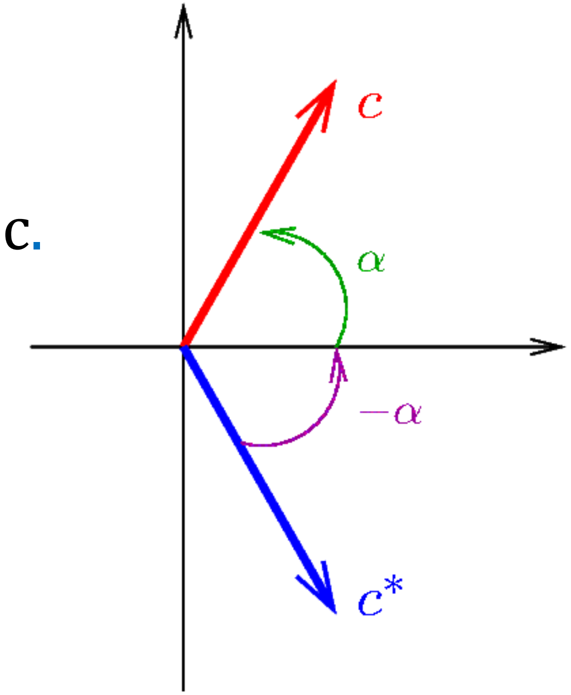
Method 2: *Multiply numerator and denominator by complex conjugate*

$$\frac{1}{1+j} = \frac{1}{1+j}\cdot\frac{1-j}{1-j} = \frac{1-j}{1^2+1^2} = \frac{1-j}{2} = \frac{1}{\sqrt{2}}\angle -45^\circ$$

We conclude that $\boldsymbol{Z_c}||\boldsymbol{Z_R} = \frac{1}{\sqrt{2}}\angle -45^\circ\ k\Omega$.

Complex Numbers—Review

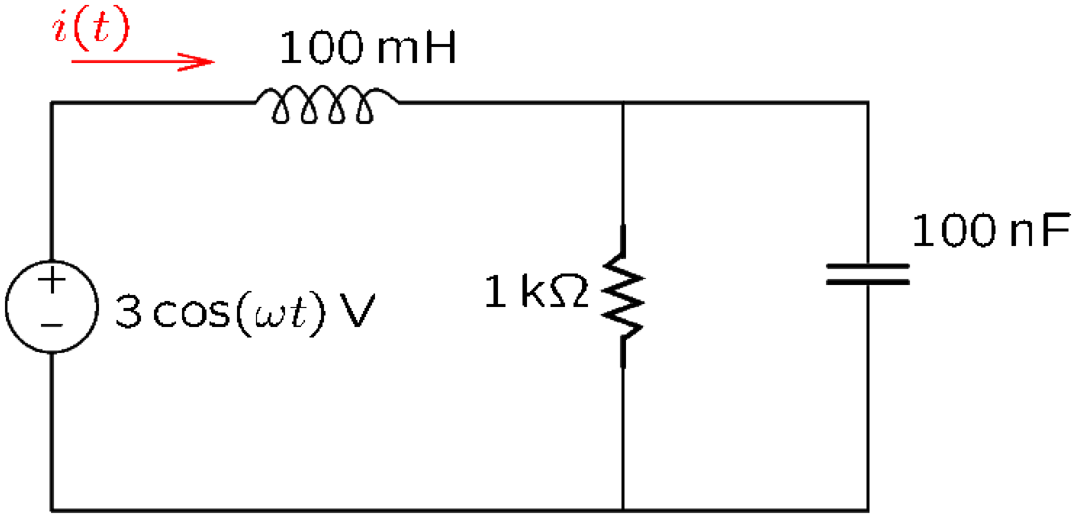
- Given a complex number $c = m + jn = |c|\angle\alpha$,
  - $|c| = \sqrt{m^2 + n^2}$ is the magnitude (aka *absolute value*) of c.
  - $m + jn$ is the rectangular form of c.
  - $|c|\angle\alpha$ is the polar form of c.
  - $c^* = m - jn = |c|\angle - \alpha$ is the complex conjugate of c.
  - Note that $c \cdot c^* = |c|^2$.



To finish our example:

$$\boldsymbol{Z} = \boldsymbol{Z_L} + (\boldsymbol{Z_C}||\boldsymbol{Z_R}) = j + \frac{1-j}{2} = \frac{1+j}{2}\ k\Omega = 707.11\angle 45^\circ\ \Omega$$

Impedance Combinations



Example: Find $i(t)$. *Assume* $f = \frac{10}{2\pi} kHz$.

- The source has a phasor voltage:

$$\boldsymbol{V_s}(t) = 3\angle\omega t \text{ V}$$

- Using the abbreviated notation:

$$\boldsymbol{V_s} = 3\angle 0° \text{ V}$$

- The total impedance $\boldsymbol{Z}$ is known from the previous example:

$$\boldsymbol{Z} = \frac{1}{\sqrt{2}}\angle 45° \, k\Omega$$

- The phasor current is

$$\boldsymbol{I} = \frac{\boldsymbol{V_s}}{\boldsymbol{Z}} = \frac{3\angle 0°}{\frac{1}{\sqrt{2}}\angle 45°} = 4.24\angle - 45° \, mA$$

- Therefore, in the time domain, $i(t) = 4.24\cos(\omega t - 45°) \; mA$.

Reactance

- Consider the rectangular form of the impedance:

$$\boldsymbol{Z} = R + jX$$

- The real part R is called resistance.
- The imaginary part X is called reactance.
- The unit of R and X is Ohms $[\Omega]$.

- Example: *For an inductor,* $\boldsymbol{Z} = j\omega L$*. The reactance is* $X = \omega L$.
- Example: *For a capacitor,* $\boldsymbol{Z} = \frac{1}{j\omega C} = -\frac{j}{\omega C}$*. The reactance is* $X = -\frac{1}{\omega C}$.

Admittance, Conductance, Susceptance

- The admittance is defined as

$$\boldsymbol{Y} = \frac{1}{\boldsymbol{Z}}$$

- Consider the rectangular form of the admittance:

$$\boldsymbol{Y} = G + jB$$

- The real part G is called conductance.
- The imaginary part B is called susceptance.
- The unit of G and B is Siemens [S] or $[\Omega^{-1}]$.

Admittance, Conductance, Susceptance

Example: Find the conductance and the susceptance of $\boldsymbol{Z} = 4 - 3j\ \Omega$**.**

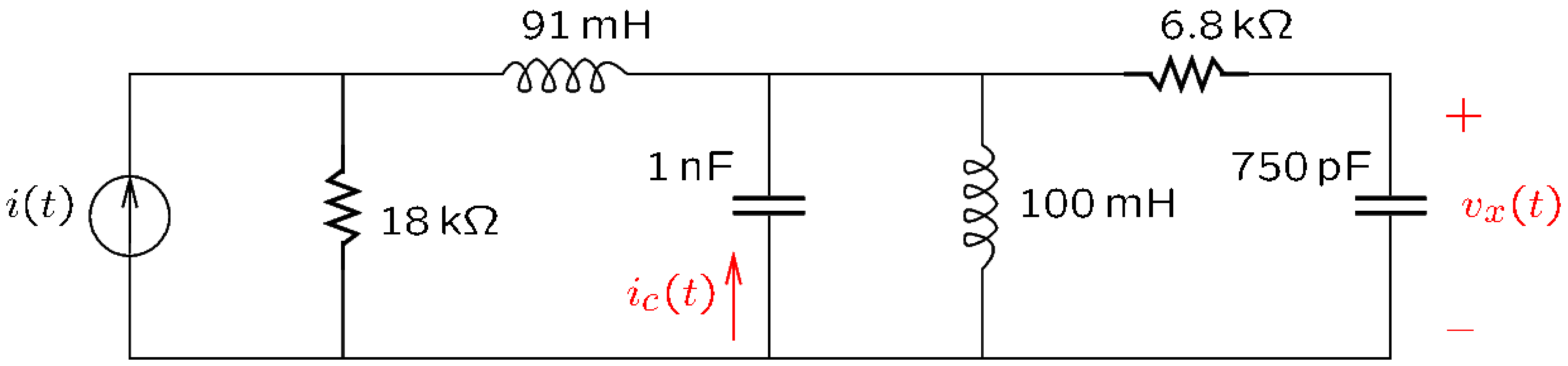
- The admittance is

$$\boldsymbol{Y} = \frac{1}{4-3j} = \frac{1}{4-3j} \cdot \frac{4+3j}{4+3j} = \frac{4+3j}{4^2 + (-3)^2} = 160 + 120j\ mS$$

- Therefore, $G = 160\ mS$ and $B = 120\ mS$.
- In this example the resistance is $R = 4\ \Omega$ and the reactance is $X = -3\ \Omega$.
- It is important to note that in general $G \neq \frac{1}{R}$ and always $B \neq \frac{1}{X}$.

Voltage and Current Division

- The methods learned for DC circuits apply also to AC circuits *in the frequency domain*.
- In particular, this is true of voltage and current division.



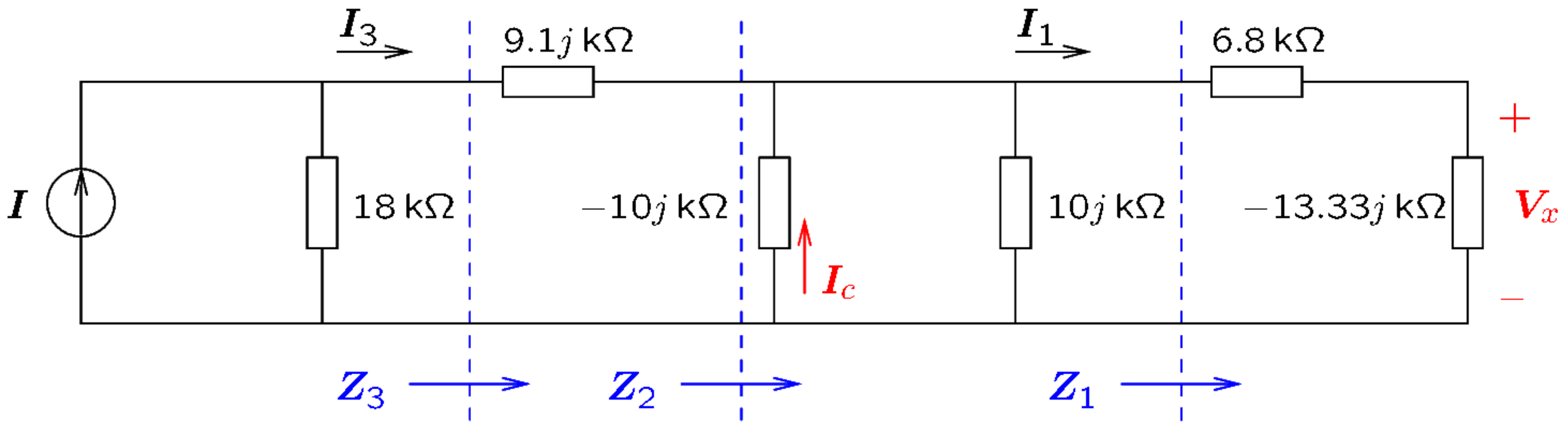
Example: Find $i_c(t)$ *and* $v_x(t)$*. Assume* $f = \frac{100}{2\pi}\,kHz$ *and* $i(t) = 10\cos(\omega t + 10°)\ \mu A$.

- Since $\omega = 2\pi f = 10^5\ rad/s$, using the formulas $\boldsymbol{Z_L} = j\omega L$ and $\boldsymbol{Z_C} = \frac{1}{j\omega C}$:

$\boldsymbol{Z_{91m}} = 9.1j\ k\Omega$, $\boldsymbol{Z_{100m}} = 10j\ k\Omega$, $\boldsymbol{Z_{1n}} = -10j\ k\Omega$, and $\boldsymbol{Z_{750p}} = -13.33j\ k\Omega$.

Voltage and Current Division

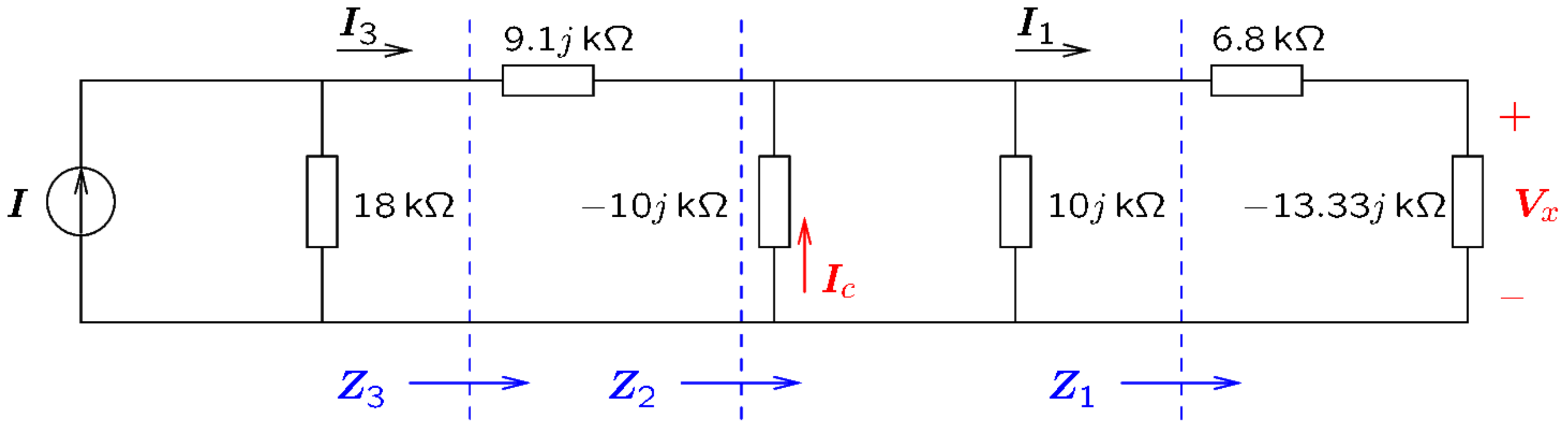
- The frequency domain circuit is shown in the figure.



The problem will be solved as follows.

- We will find first $\boldsymbol{Z_1}$, $\boldsymbol{Z_2}$, and $\boldsymbol{Z_3}$.
- Next, current division will be used to find $\boldsymbol{I_3}$, $\boldsymbol{I_c}$, and $\boldsymbol{I_1}$.
- Next, Ohm's law will be used to find $\boldsymbol{V_x} = -13.33j\ k\Omega \cdot \boldsymbol{I_1}$.
- The results will be converted to the time domain.

Voltage and Current Division



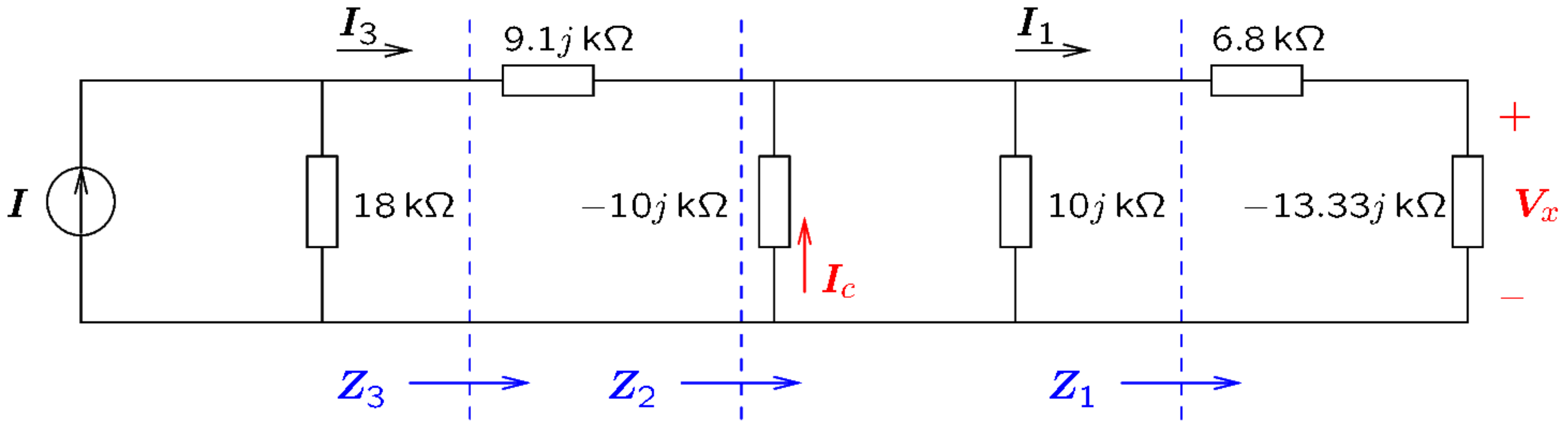
$$\boldsymbol{Z_1} = 6.8 - 13.33j\ k\Omega$$

$$\boldsymbol{Z_2} = \frac{1}{\frac{1}{-10j\ k\Omega} + \frac{1}{10j\ k\Omega} + \frac{1}{\boldsymbol{Z_1}}} = \boldsymbol{Z_1}$$

$$\boldsymbol{Z_3} = 9.1j\ k\Omega + \boldsymbol{Z_2} = 6.8 - 4.23j\ k\Omega$$

Since $i(t) = 10\cos(\omega t + 10°)\ \mu A$, it follows that $\boldsymbol{I} = 10\angle 10°\ \mu A$.

Voltage and Current Division

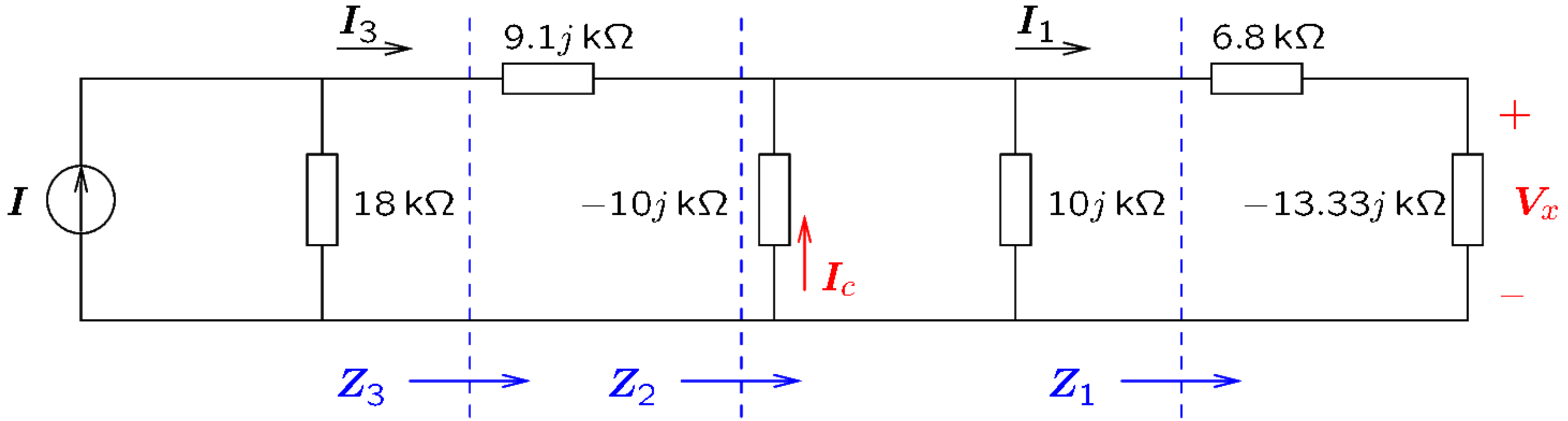


$$\boldsymbol{I_3} = \boldsymbol{I}\frac{\frac{1}{\boldsymbol{Z_3}}}{\frac{1}{18\ k\Omega}+\frac{1}{\boldsymbol{Z_3}}} = 10\angle 10^\circ \mu A\,\frac{18\ k\Omega}{18\ k\Omega+\boldsymbol{Z_3}} = 10\angle 10^\circ \mu A\,\frac{18}{25.16\angle-9.68^\circ} = 7.15\angle 19.68^\circ\ \mu A$$

$$\boldsymbol{I_1} = \boldsymbol{I_3}\frac{\frac{1}{\boldsymbol{Z_1}}}{\frac{1}{-10j\ k\Omega}+\frac{1}{10j\ k\Omega}+\frac{1}{\boldsymbol{Z_1}}} = \boldsymbol{I_3} \Rightarrow \boldsymbol{V_x} = -13.33j\ k\Omega\cdot\boldsymbol{I_1} = 95.37\angle-70.32^\circ\ mV$$

$$\Rightarrow v_x(t) = 95.37\cos(\omega t-70.32^\circ)\ \ mV$$

Voltage and Current Division



$$\boldsymbol{I_c} = -\boldsymbol{I}\,\frac{\dfrac{1}{-10j\ k\Omega}}{\dfrac{1}{-10j\ k\Omega}+\dfrac{1}{10j\ k\Omega}+\dfrac{1}{\boldsymbol{Z_1}}} = 10\angle 10^\circ \mu A\,\frac{\boldsymbol{Z_1}}{10j\ k\Omega} = 14.96\angle -142.97^\circ\ \mu A$$

$$\Rightarrow i_c(t) = 14.96\cos(\omega t - 142.97^\circ)\ \ \mu A$$

Remarks

- In the previous example, the current source had $i(t) = 10\cos(\omega t + 10°)\ \mu A$, while the branch current $i_c(t)$ was $i_c(t) = 14.96\cos(\omega t - 142.97°)\ \mu A$.
- *In AC, currents/voltages do not have to be smaller than the value of the current/voltage source that causes them.*
- The previous example also illustrates that *an impedance can be infinite:*

$$(-10j\ k\Omega)||(10j\ k\Omega) = \frac{1}{\frac{1}{-10j} + \frac{1}{10j}} = \frac{1}{0} = \infty$$

- This impedance is infinite because the $1\ nF$ capacitor and the $100\ mH$ inductor cancel each other at the given frequency.