Phasors

Representing Sinusoidal Voltages and Currents in the Frequency Domain

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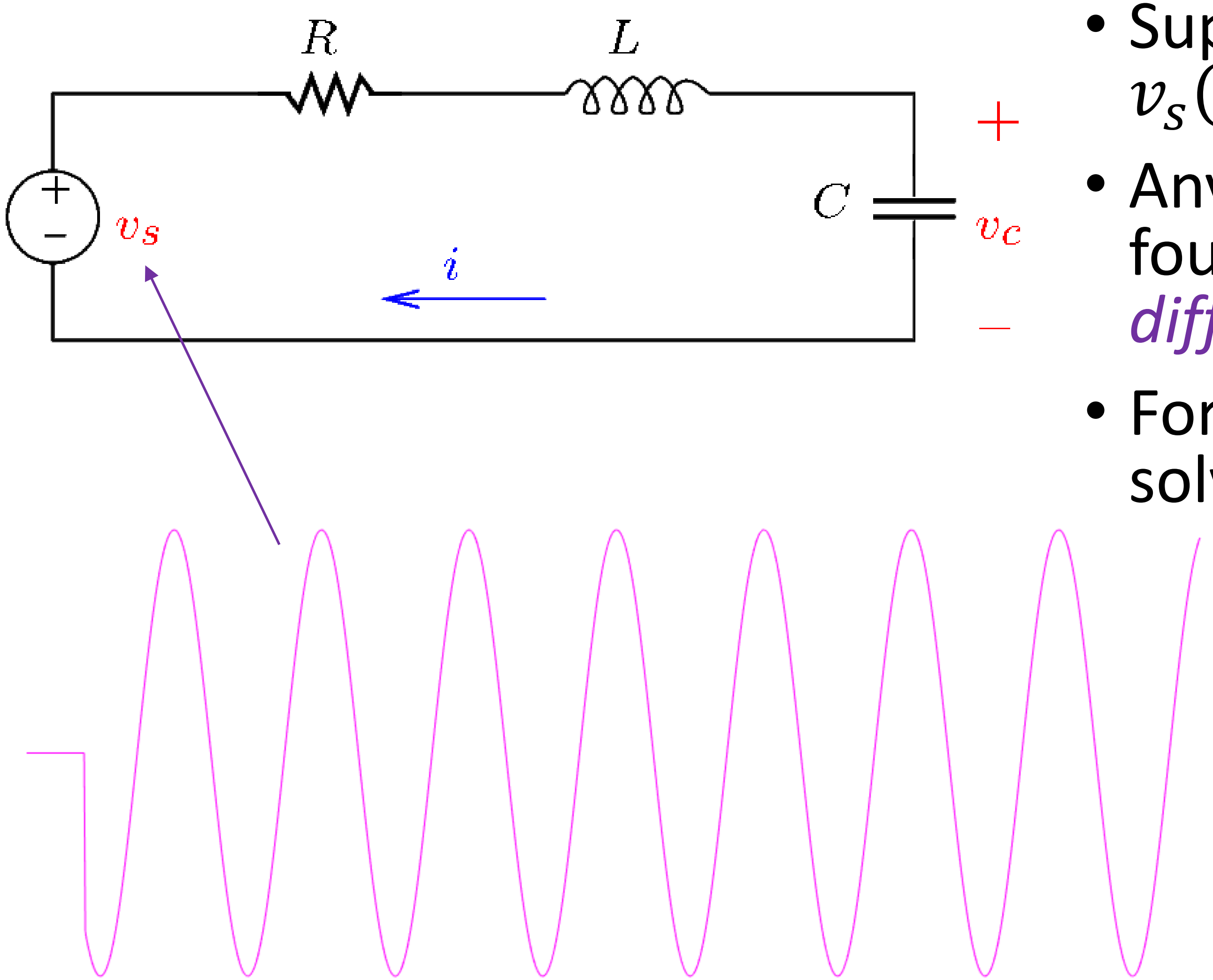
Introduction



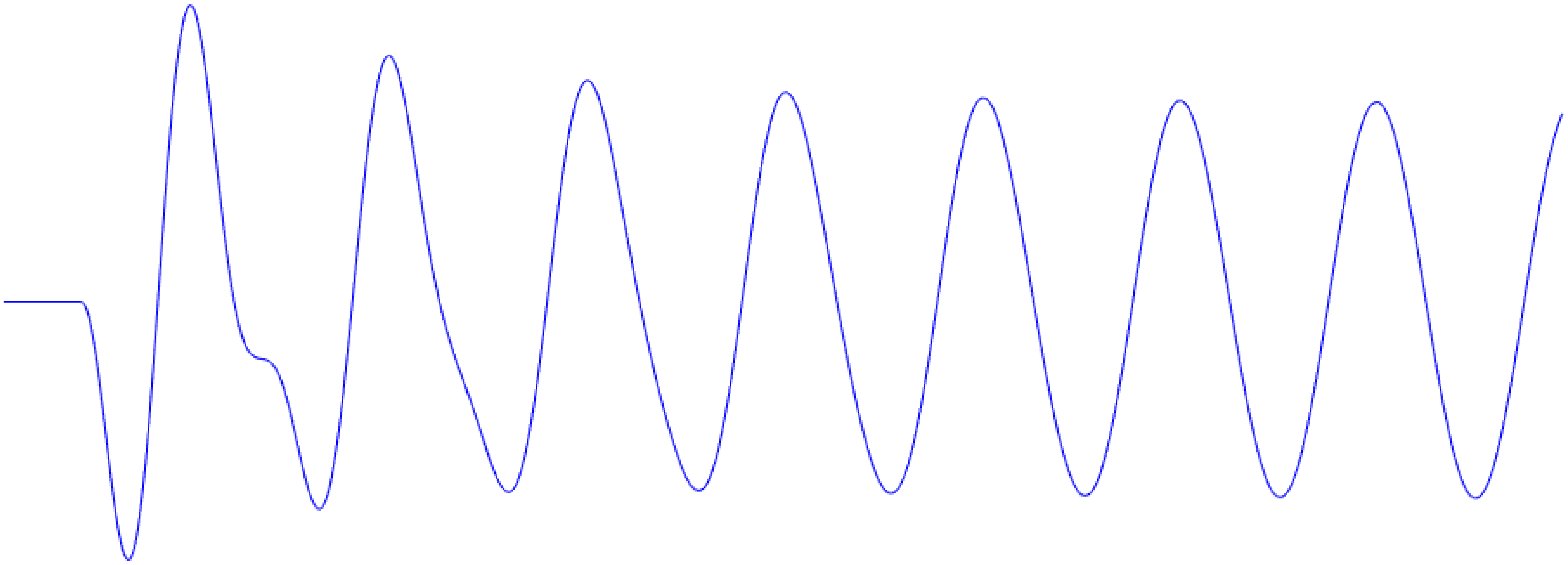
- Suppose that a sinusoidal voltage $v_s(t)$ is applied at time t_0.
- Any voltage or current could be found by solving a system of *differential equations*.
- For example, $v_c(t)$ is found by solving the system of equations:

$$i = C\frac{dv_c}{dt}$$

$$iR + L\frac{di}{dt} + v_c = v_s$$

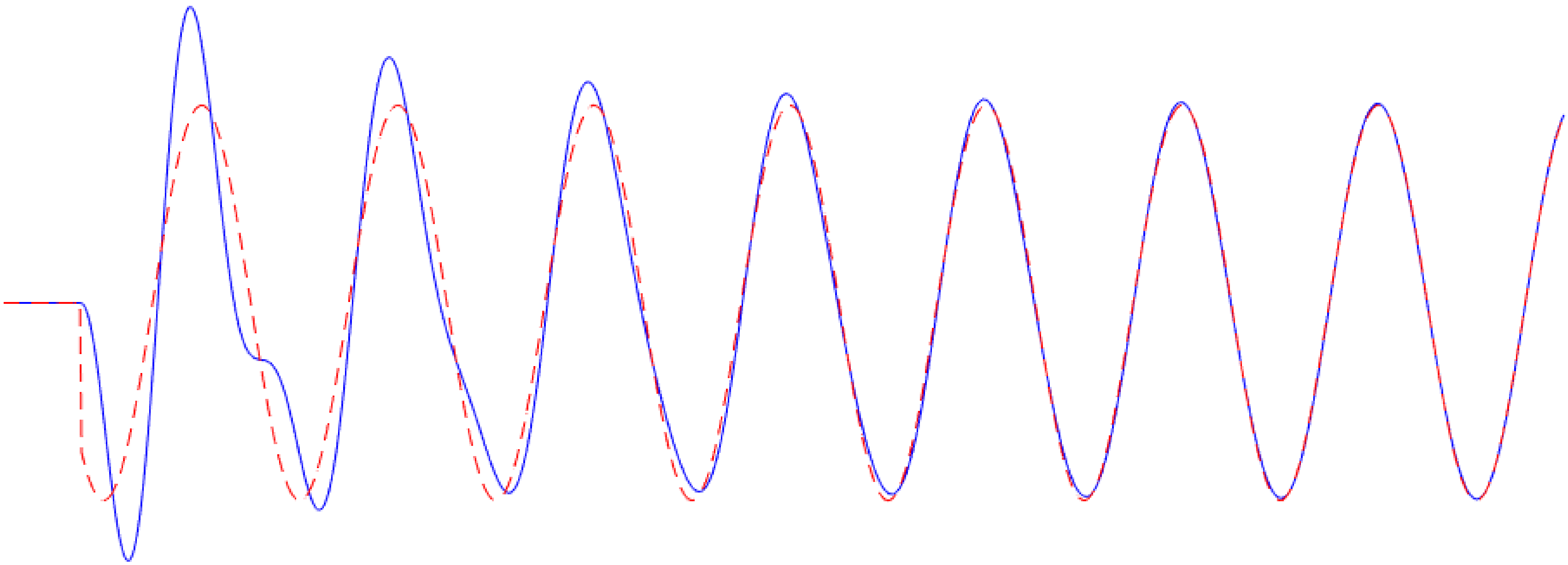
Introduction

By solving the differential equations, we obtain $v_c(t)$ having the following graph.



Introduction

- Note that $v_c(t)$ converges to the red curve.
- The red curve is called *steady-state response*.



Introduction

- Often, we do not care about transients, but we only need the steady-state.
- The steady-state response can be obtained without solving differential equations.
- To find the steady-state response, we will use this simpler method:
  - Convert the circuit to the frequency domain.
  - Solve for the unknowns in the frequency domain.
  - Convert the unknowns back to the time domain.

The Frequency Domain

- Up to this point we have worked in the *time domain*.
- Let $m(t) = M\cos(\omega t + \alpha)$ be a current or a voltage.
- We say that $m(t)$ is in the *time-domain*.
- The frequency domain representation of $m(t)$ is the complex-valued function

$$\boldsymbol{M}(t) = Me^{i(\omega t + \alpha)}$$

- Note that $Me^{i(\omega t+\alpha)}$ is the complex number of magnitude M and angle $\omega t + \alpha$.
- We also write $Me^{i(\omega t+\alpha)}$ as $M\angle\omega t + \alpha$.

Example:

$$10\angle 30^\circ = 10e^{i30^\circ} = 10\cos 30^\circ + i10\sin 30^\circ = 5\sqrt{3} + i5.$$

The Frequency Domain—Notation

- Note that $i = 1\angle 90^\circ = \sqrt{-1}$ is the unit vector in the direction of the imaginary axis.
- From now on, to avoid confusion with currents, j will be used in the place of i.
- So $j = 1\angle 90^\circ = \sqrt{-1}$.

Example: $h = 3 + 4j$ is the complex number of real part 3 and imaginary part 4.

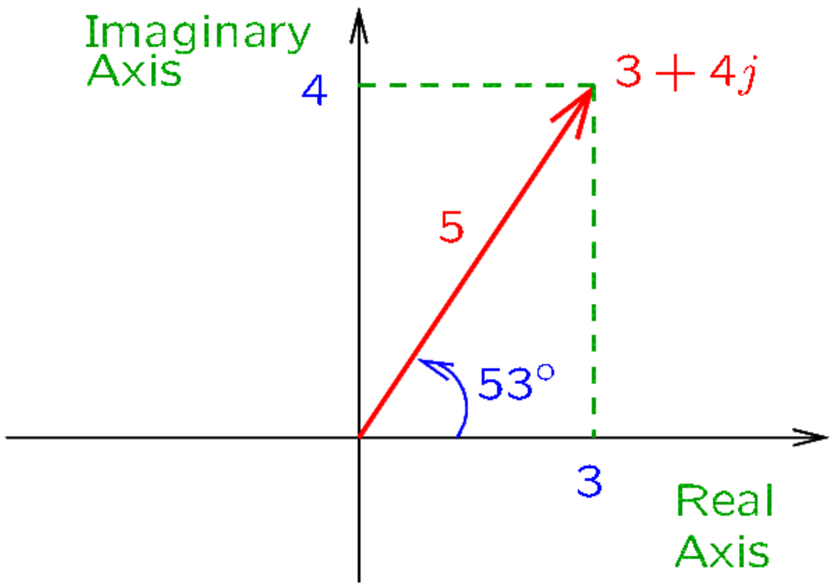
That is, $Real(h) = 3$ and $Imag(h) = 4$.

In the complex plane, h is represented by

a vector of length $|h| = \sqrt{3^2 + 4^2} = 5$,

at an angle $\arg(h) = \tan^{-1}\frac{4}{3} = 53.13^\circ$.

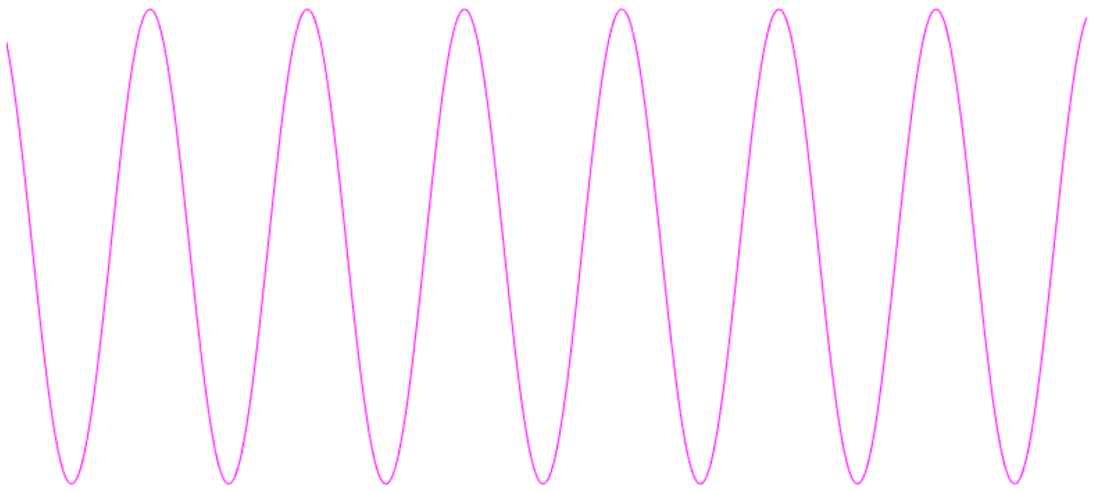
We write that $h = 3 + 4j = 5\angle 53.15^\circ$.



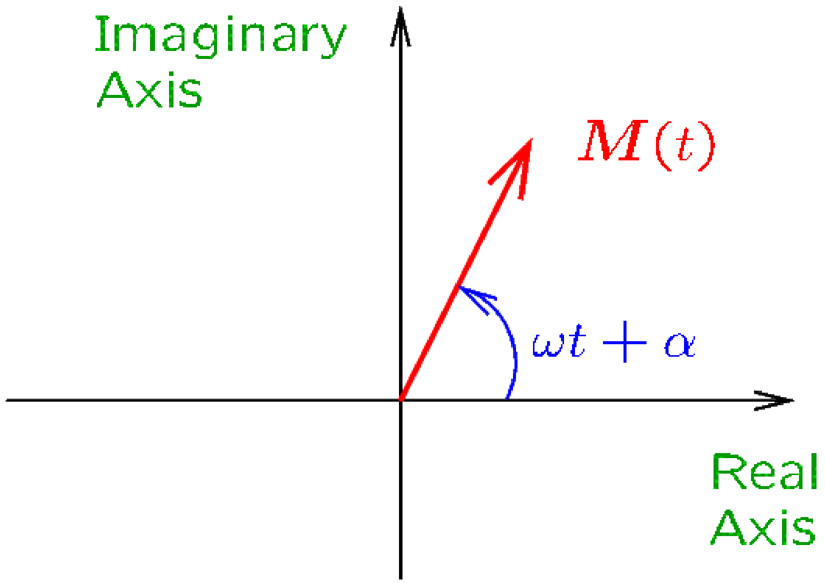
The Frequency Domain

- In the time domain, the signal $m(t) = M\cos(\omega t + \alpha)$ is a sinusoidal function.
- In the frequency domain, the signal is a vector $\boldsymbol{M}(t) = M\angle\omega t + \alpha$ that rotates counterclockwise in the complex plane at an angular velocity ω.
- Such rotating vectors are called **PHASORS**.

TIME DOMAIN



FREQUENCY DOMAIN



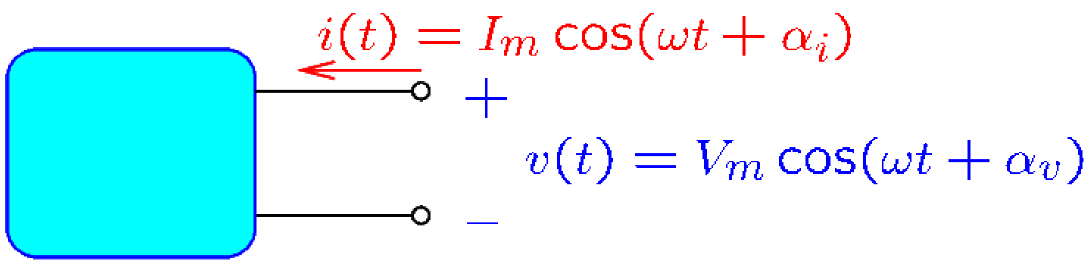
Impedance

- Assume a network with two terminals that has *no independent sources*.
- A voltage $v(t) = V_m \cos(\omega t + \alpha_v)$ will result in a sinusoidal steady-state current of the same frequency: $i(t) = I_m \cos(\omega t + \alpha_i)$.
- In the frequency domain, these correspond to the phasors

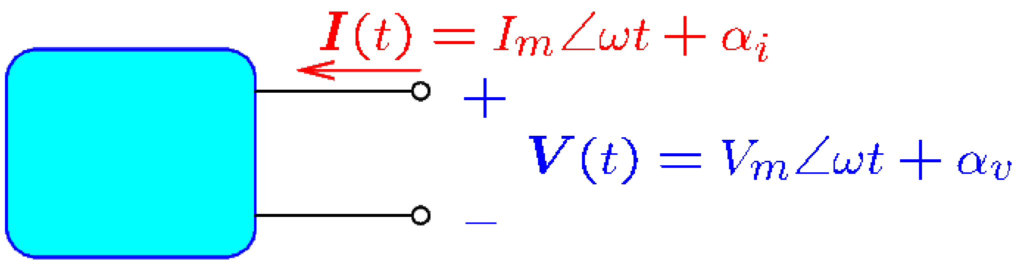
$$\boldsymbol{V}(t) = V_m e^{j\omega t + j\alpha_v} \text{ and } \boldsymbol{I}(t) = I_m e^{j\omega t + j\alpha_i}.$$

- For a linear network, the ratio $\boldsymbol{Z} = \boldsymbol{V}(t)/\boldsymbol{I}(t)$ is a constant called **impedance**.

TIME DOMAIN



FREQUENCY DOMAIN



$$\boldsymbol{Z} = \frac{\boldsymbol{V}(t)}{\boldsymbol{I}(t)} = \frac{V_m}{I_m} \angle \alpha_v - \alpha_i$$

Impedance

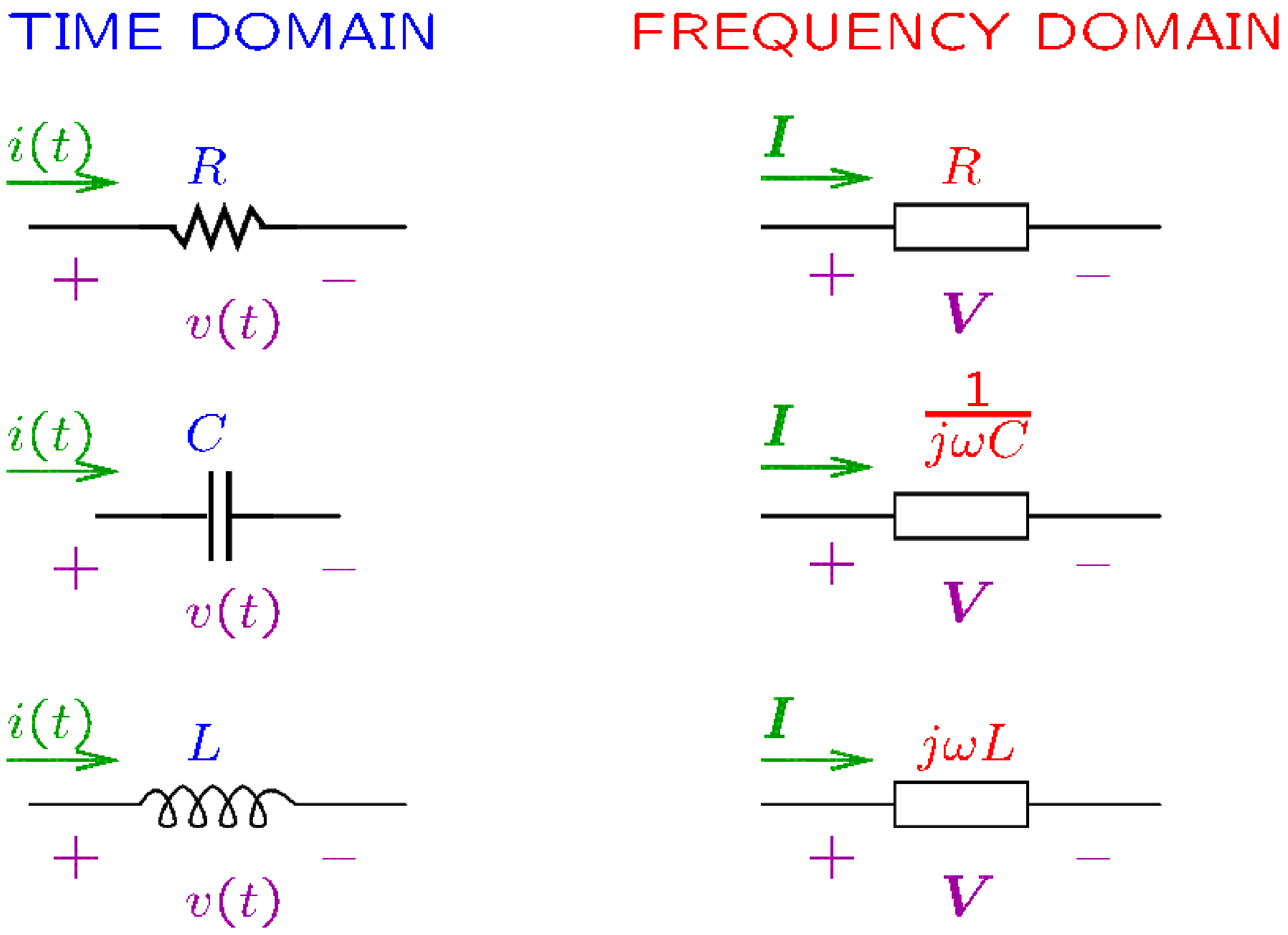
- Since $\boldsymbol{V}(t) = V_m e^{j\omega t + j\alpha_v}$ and $\boldsymbol{I}(t) = I_m e^{j\omega t + j\alpha_i}$, impedance is independent of time:

$$\boldsymbol{Z} = \frac{\boldsymbol{V}(t)}{\boldsymbol{I}(t)} = \frac{V_m}{I_m} e^{j(\alpha_v - \alpha_i)}$$

- *The impedance is a constant, not a phasor!*
- *Only voltages and currents are phasors.*
- The impedance unit is Ohms [Ω].
- *Impedance resembles resistance; it could be viewed as an "AC resistance".*

Impedance of Common Components

Resistors, capacitors, and inductors are represented in the frequency domain by their impedance.



Phasors—Abbreviated Notation

- Assume all sources of a circuit have the same frequency.
- Every phasor $\boldsymbol{N}(t) = N_m\angle\omega t + \alpha_n$ will differ from any other phasor $\boldsymbol{P}(t) = P_m\angle\omega t + \alpha_p$ only in magnitude and angle.
- The term ωt is common for all phasors.
- From now on, we will drop the ωt term.
- $\boldsymbol{N}(t) = N_m\angle\omega t + \alpha_n$ will be abbreviated $\boldsymbol{N} = N_m\angle\alpha_n$.

Examples

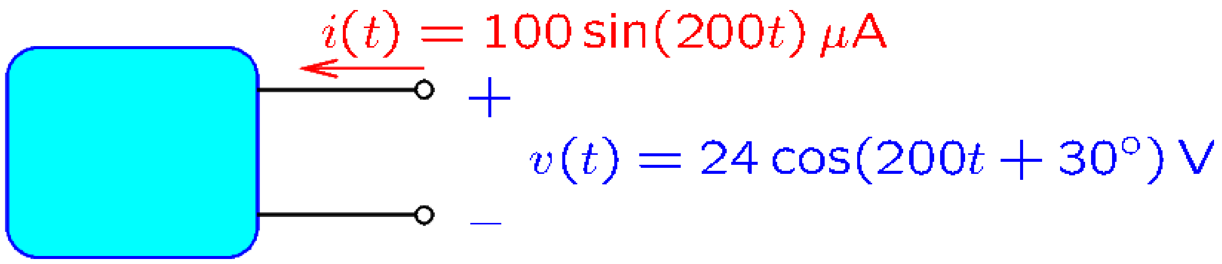
Write $v_x(t) = 3\,cos(100t + 45°)$ V in the frequency domain.

$\boldsymbol{V}_x = 3\angle 45°$ V.

Write $I_x = 50\angle - 45°$ mA in the time domain.

$i_x(t) = 50\cos(\omega t - 45°)$ mA.

Find the impedance of the network.



- $i(t) = 100\cos(200t - 90°)$ μA.
- $\boldsymbol{I} = 100\angle - 90°$ μA.
- $\boldsymbol{V} = 24\angle 30°$ V.
- $\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{24\angle 30°\ \text{V}}{100\angle -90°\ \mu\text{A}} = 240\angle 120°\ \text{k}\Omega$.

Example

Assume $v_1(t) = 3\cos(200t + 30°)$ V and $v_2(t) = 30\cos(200t - 60°)$ V . Write the total voltage $v_t(t) = v_1(t) + v_2(t)$ as a sinusoidal function of time.

- $\boldsymbol{V}_1 = 3\angle 30° = 3\cos(30°) + j3\sin(30°) = 1.5\sqrt{3} + 1.5j$ V.
- $\boldsymbol{V}_2 = 30\angle -60° = 30\cos(-60°) + j30\sin(-60°) = 15 - j15\sqrt{3}$ V.
- $\boldsymbol{V}_t = \boldsymbol{V}_1 + \boldsymbol{V}_2 = 17.6 - 24.48j = \sqrt{17.6^2 + (-24.48)^2}\angle\alpha = 30.15\angle\alpha$ V.
- The angle $\alpha = \tan^{-1}\left(-\frac{24.48}{17.6}\right) + k \cdot 180°$,

  where k could be 0 or ± 1, depending on the quadrant.
- In this case $\alpha = -54.29° + k \cdot 180°$.
- The angle is in a quadrant with $k = 0$.
- Hence, $\boldsymbol{V}_t = 30.15\angle -54.29°$ V.
- The final answer is

  $v_t(t) = 30.15\cos(200t - 54.29°)$ V.