Frequency-Domain Measurements

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See https://mviordache.name/EEGR2051 for more information.

Fourier Series

Any periodic signal $y(t)$ of frequency ω can be represented as a sum of sinusoidal terms of frequency $\omega, 2\omega, 3\omega$, ...

$$y(t) = c_0 + c_1 \sin(\omega t + \phi_1) + c_2 \sin(2\omega t + \phi_2) + c_3 \sin(3\omega t + \phi_3) + \cdots$$

where:

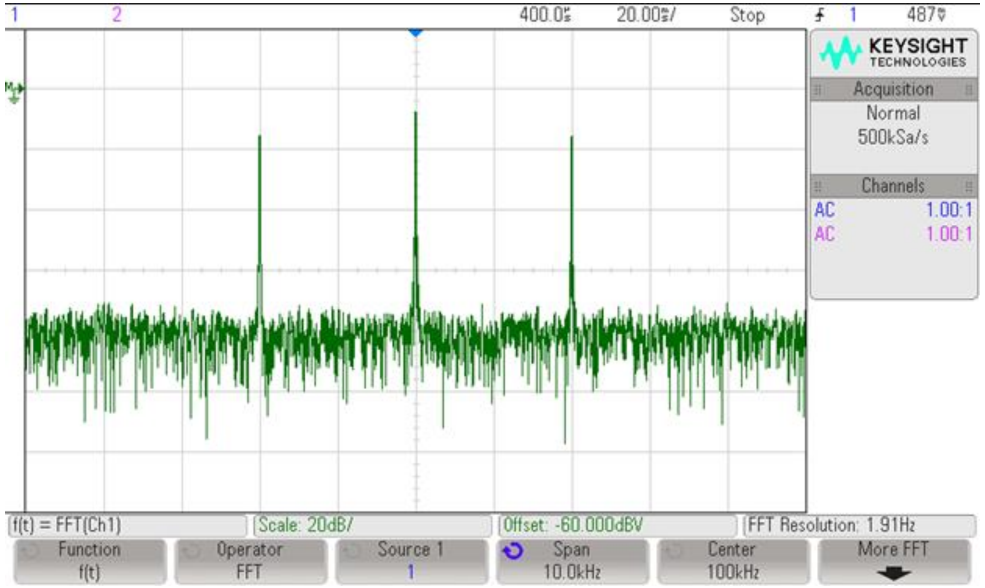
- c_0 is the *DC component* of $y(t)$.
- $c_1 \sin(\omega t + \phi_1)$ is the *fundamental* component of $y(t)$.
- For all $n = 1, 2, 3, \ldots,$ $c_n \sin(n\omega t + \phi_n)$ is the *harmonic of order* n.

Fourier Series

- Harmonics can be measured with a *spectrum analyzer*.

Rohde-Schwarz spectrum analyzer. Image from wikipedia.org.

- Many digital oscilloscopes include an *FFT* option allowing to measure the harmonics of a signal.



f(t) = FFT(Ch1)

Scale: 20dB/

Offset: -60.000dBV

FFT Resolution: 1.91Hz

Acquisition: Normal, 500kSa/s

Fourier Series

- Let $T = \frac{2\pi}{\omega}$ be the period of $y(t)$.
- Harmonics can be calculated with the equations

$$c_0 = \frac{1}{T}\int_0^T y(t)dt\,, \qquad c_n = \sqrt{a_n^2 + b_n^2},$$

$$\cos\phi_n = \frac{b_n}{c_n}, \qquad \sin\phi_n = \frac{a_n}{c_n}$$

where

$$a_n = \frac{2}{T}\int_0^T y(t)\cos(n\omega t)\,dt\,, \qquad b_n = \frac{2}{T}\int_0^T y(t)\sin(n\omega t)\,dt$$

Example

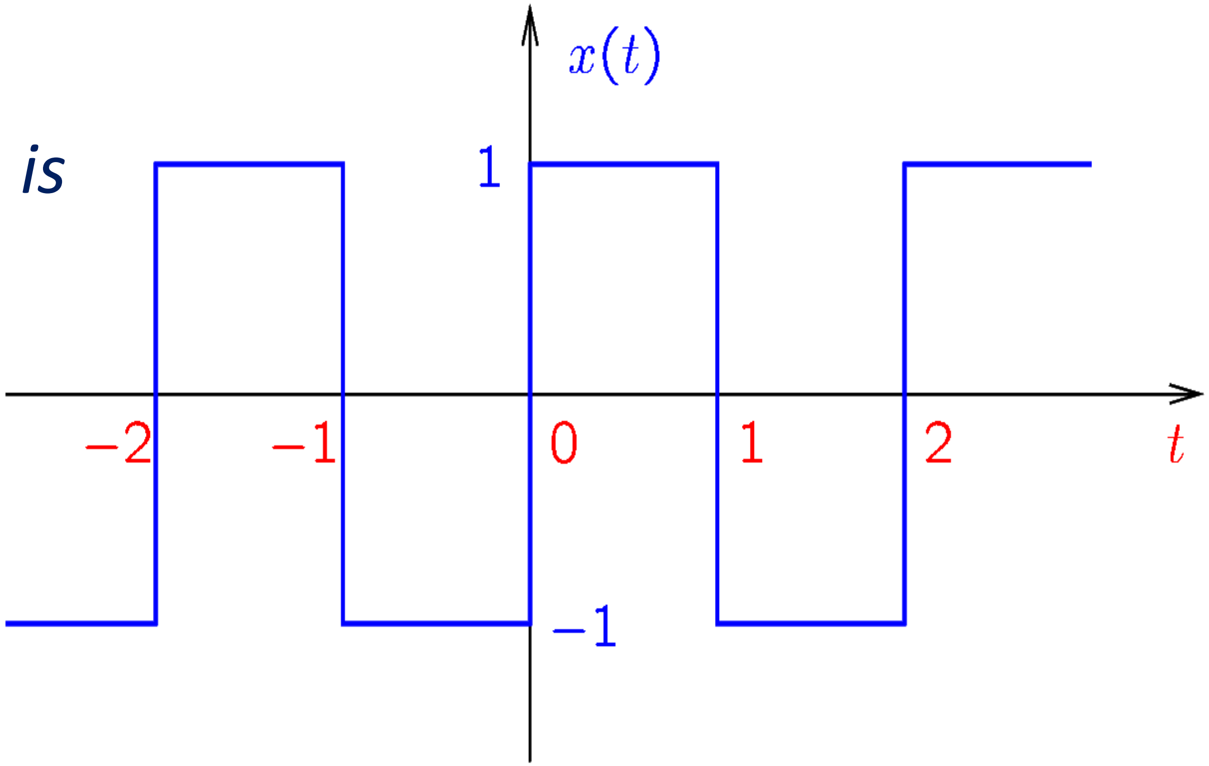
- *The period of the shown waveform is*

  $$T = 2.$$

- *The angular frequency of the fundamental is therefore,*

  $$\omega = \frac{2\pi}{T} = \pi.$$

- *It can be shown that*



$$x(t) = \frac{4}{\pi} sin(\omega t) + \frac{4}{3\pi} sin(3\omega t) + \frac{4}{5\pi} sin(5\omega t) + \cdots$$

Example (continued)

- *Since*

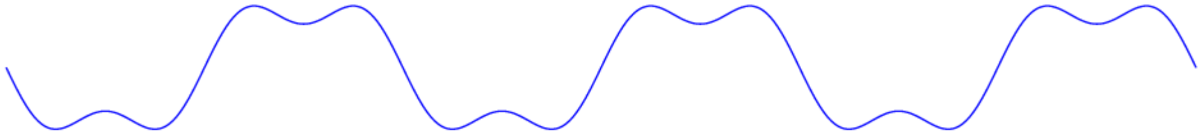
$$x(t) = \frac{4}{\pi} sin(\omega t) + \frac{4}{3\pi} sin(3\omega t) + \frac{4}{5\pi} sin(5\omega t) + \cdots$$

  - *The average of* $x(t)$ *is* $c_0 = 0$.
  - *The fundamental of* $x(t)$ *has the amplitude* $c_1 = \frac{4}{\pi}$.
  - *The second harmonic of* $x(t)$ *has the amplitude* $c_2 = 0$.
  - *The third harmonic of* $x(t)$ *has the amplitude* $c_3 = \frac{4}{3\pi}$.

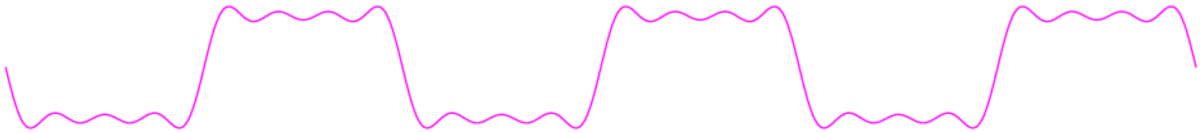
Example (continued)

- *The more harmonics are included, the better the approximation!*

$$\frac{4}{\pi}\cos(\omega t) + \frac{4}{3\pi}\cos(3\omega t) \rightarrow$$



$$\frac{4}{\pi}\cos(\omega t) + \cdots + \frac{4}{7\pi}\cos(7\omega t) \rightarrow$$



$$\frac{4}{\pi}\cos(\omega t) + \cdots + \frac{4}{11\pi}\cos(11\omega t) \rightarrow$$

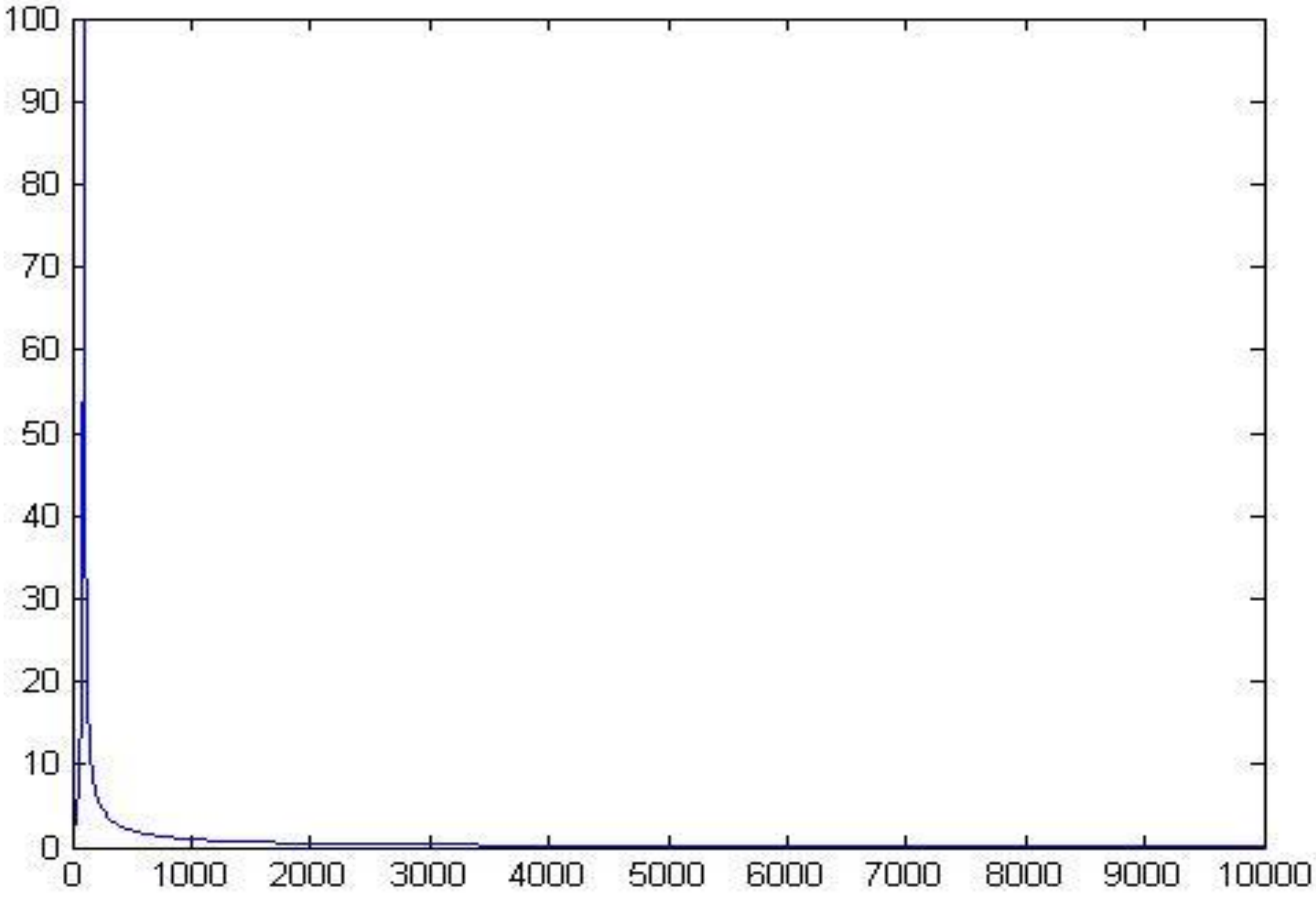
$$\frac{4}{\pi}\cos(\omega t) + \cdots + \frac{4}{15\pi}\cos(15\omega t) \rightarrow$$

Logarithmic Scale

- The figure shows the graph of Y versus x in blue.

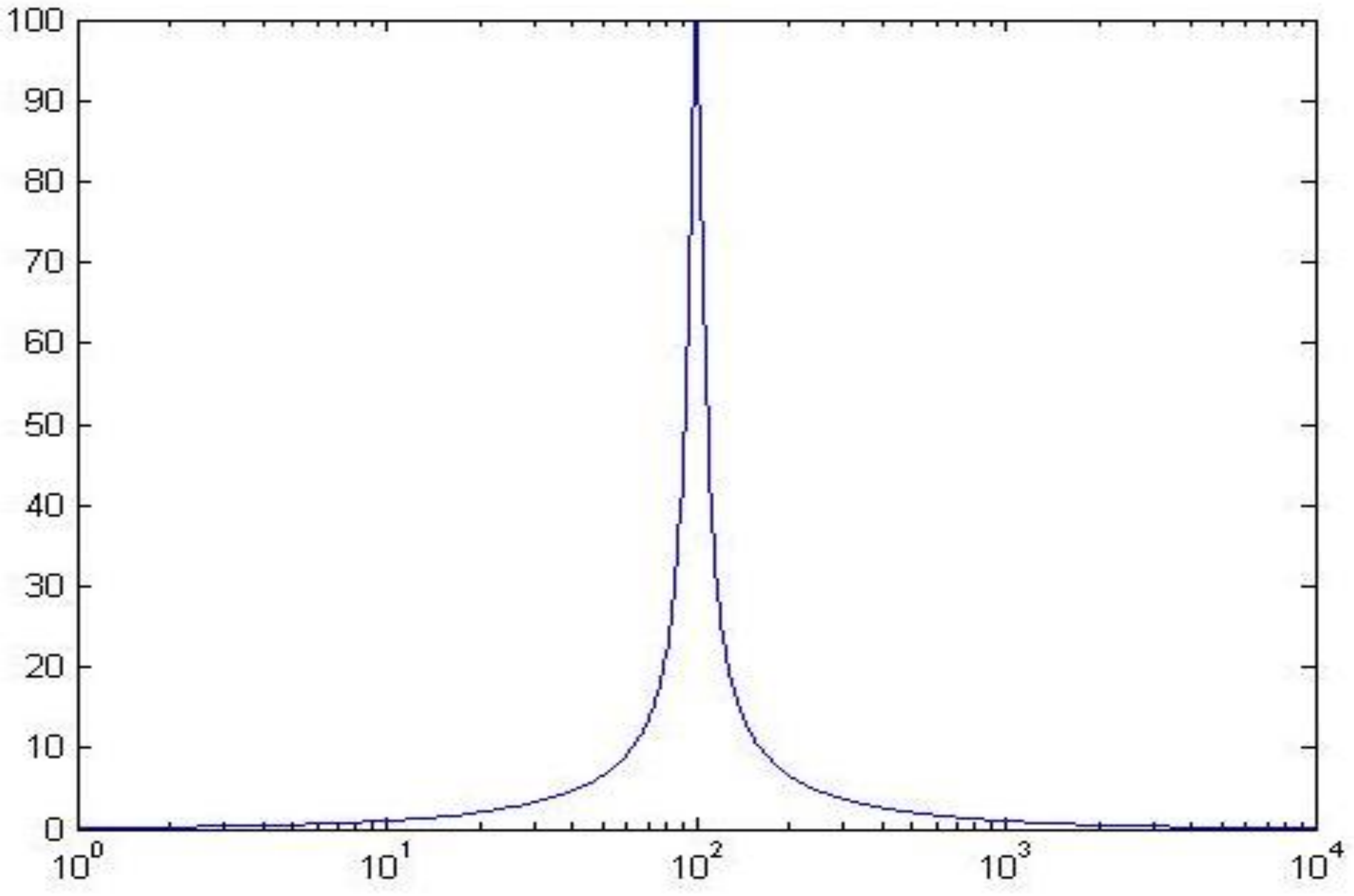
- The values of Y should be read for $x \ = 10, \ \ 100, \ 1000, \ \text{and} \ 10000.$

- Without question, the values of Y cannot be found precisely by looking at this graph.



Logarithmic Scale

- With a *logarithmic scale* on x, the graph shows Y versus $\log_{10} x$ and labels the x-axis in terms of the values of x (not in terms of $\log_{10} x$.)
- Now the x-axis shows clearly where $x = 10$ and where $x = 100$.
- However, the value of Y at $x = 10000$ cannot be read precisely; all we can tell is that it is close to zero.
- This is because the maximum and minimum values of Y differ by orders of magnitude.

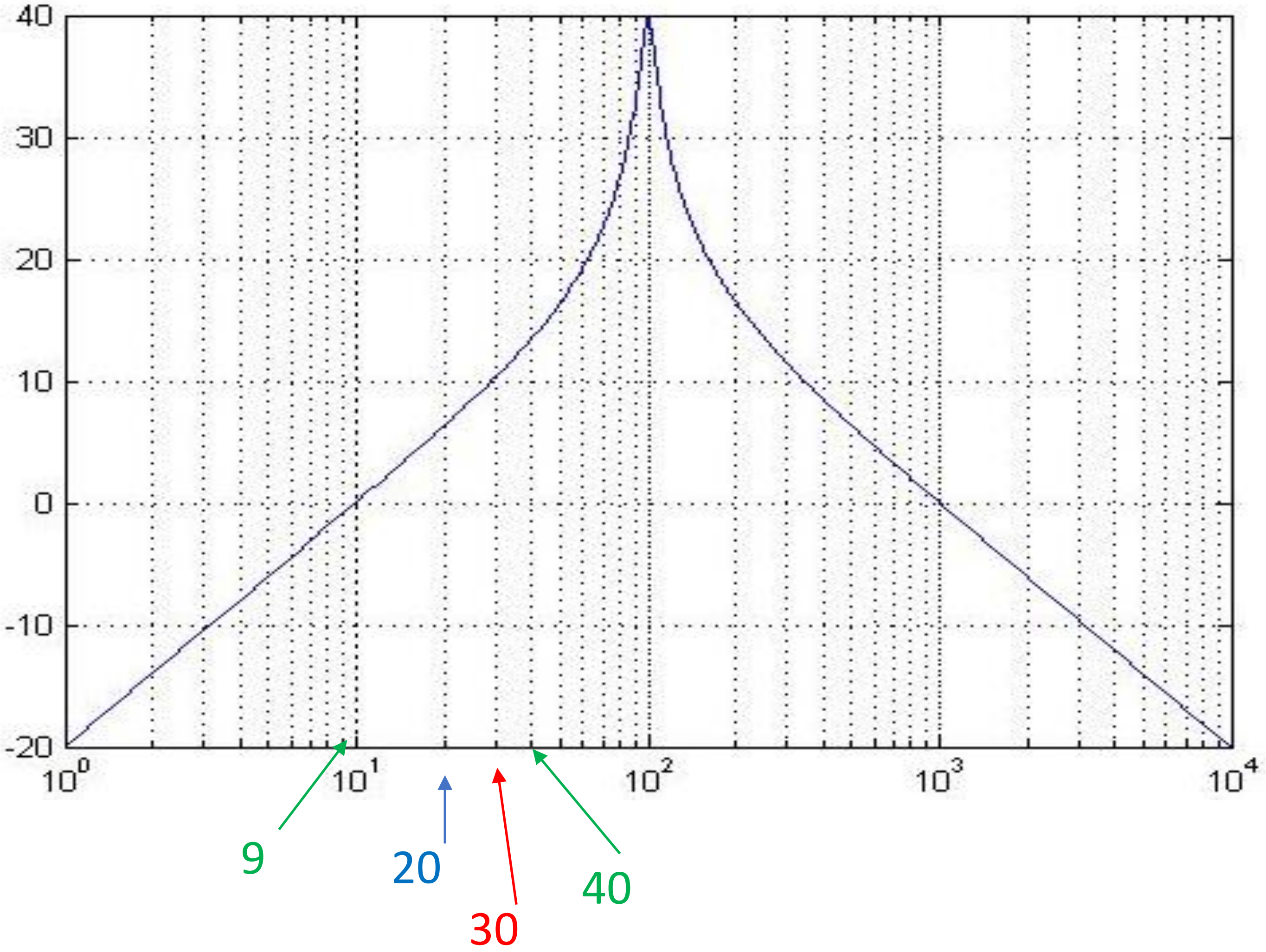


Logarithmic Scale

- The plot in the figure shows Y in decibels versus x, with logarithmic scale on the x-axis.
- With a logarithm on the y-axis, the low values of Y are now clearly seen.
- Note that a voltage Y in volts can be converted to decibels with the formula:

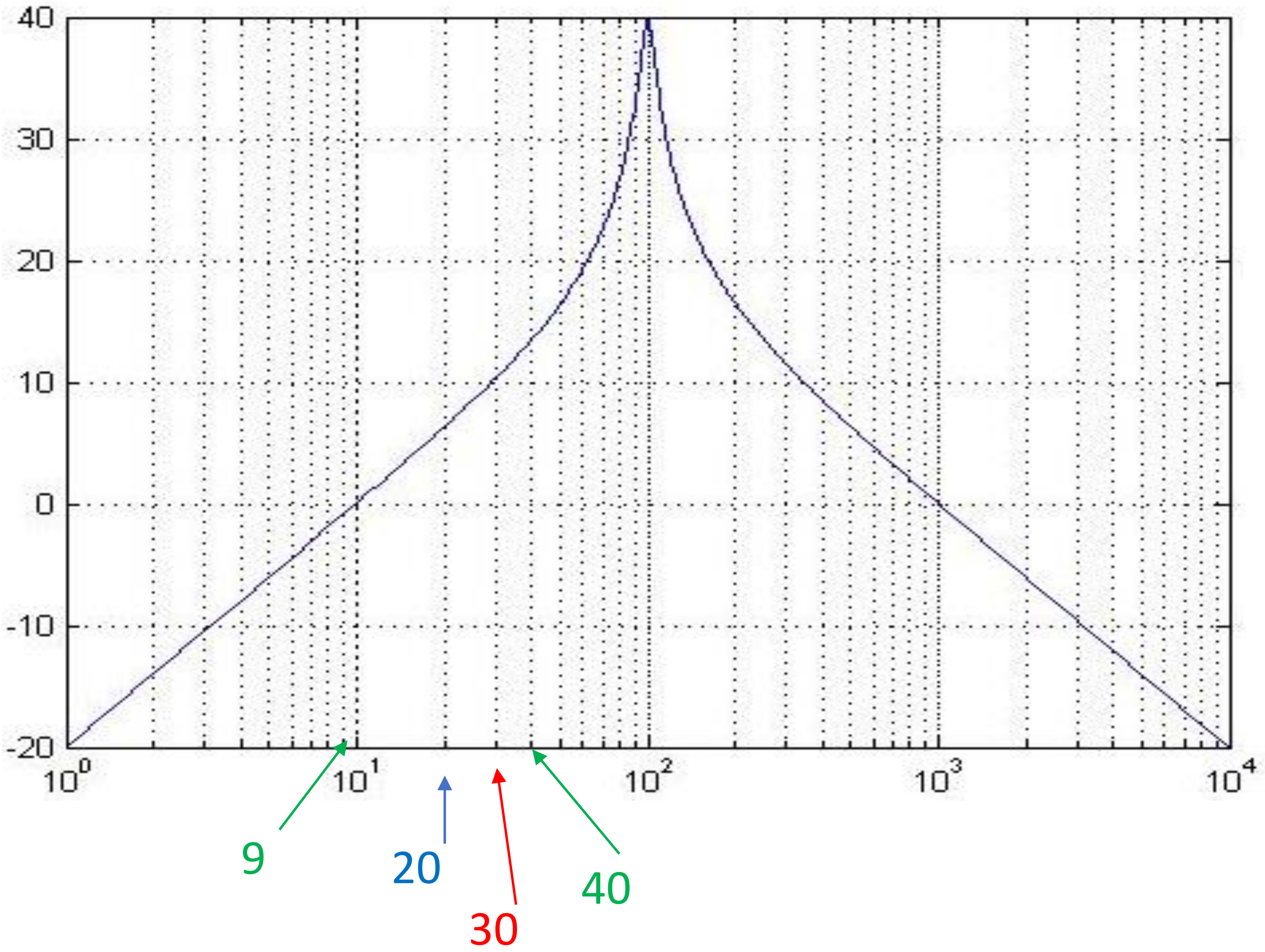
$$Y_{dB} = 20\log_{10} Y$$

- For example, if $Y = 10\ V$ rms, then $Y_{dB} = 20\log_{10} 10 = 20\ dBV$, where the V suffix of dBV indicates that Y is in volts.



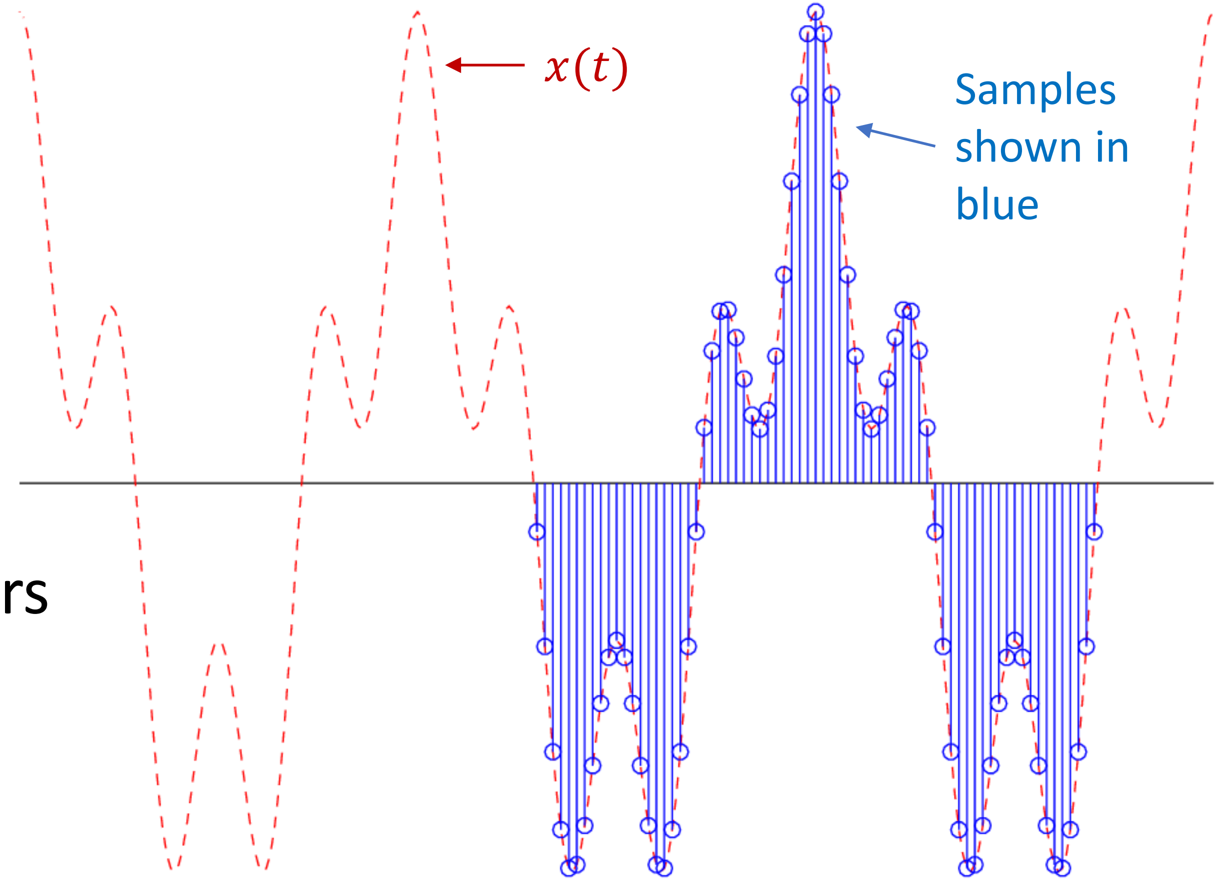
Example

- The vertical gridlines help us read values that are not powers of 10.
- For $x = 10$,
  $Y_{dB} = 0\ dBV \Rightarrow Y = 1\ V$.
- For $x = 10^2$,
  $Y_{dB} = 40\ dBV \Rightarrow Y = 100\ V$.
- For $x = 10^3$,
  $Y_{dB} = 0\ dBV \Rightarrow Y = 1\ V$.
- For $x = 10^4$,
  $Y_{dB} = -20\ dBV \Rightarrow Y = 0.1\ V$.
- For $x = 2000$,
  $Y_{dB} = -6\ dBV \Rightarrow Y = 0.5\ V$.



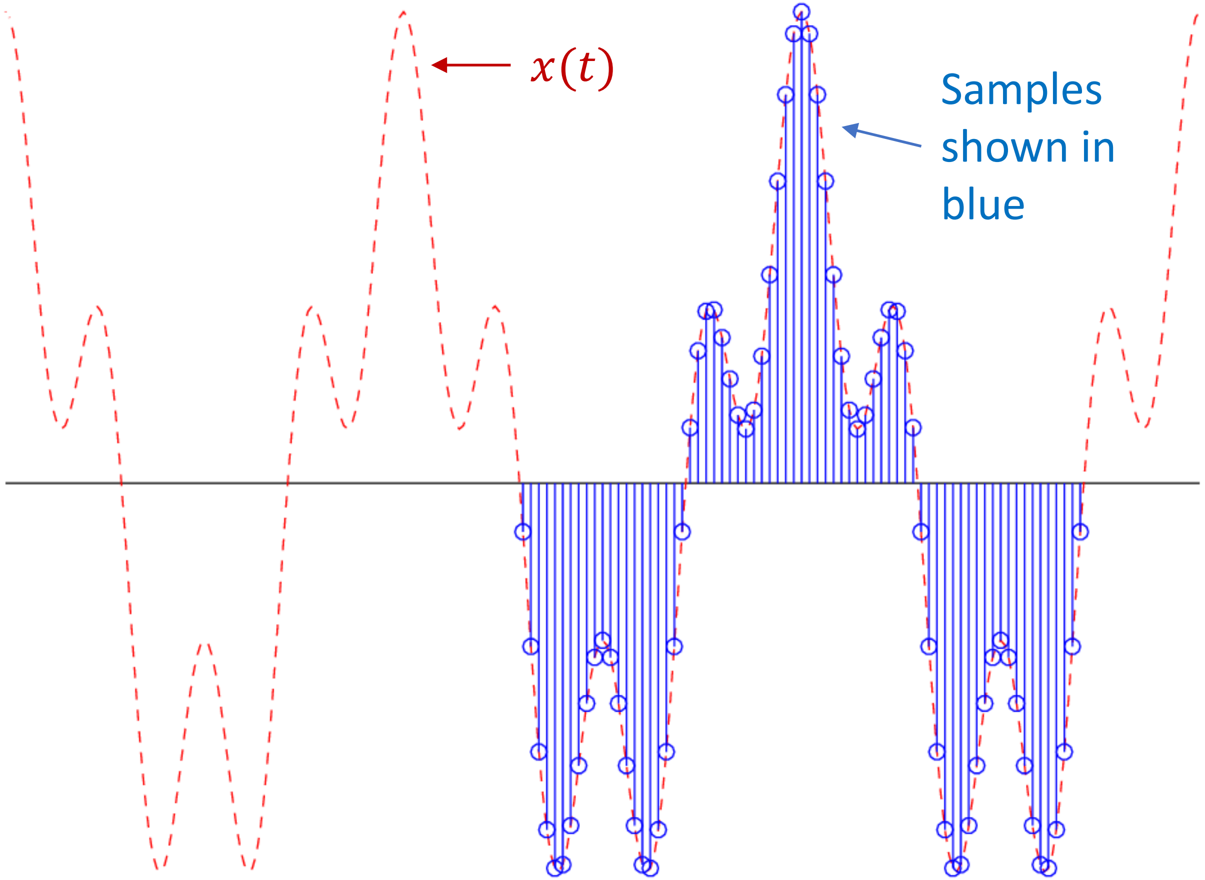
FFT

- Each signal can be decomposed into a sum of sines.
- The *Fast Fourier Transform* (FFT) algorithm provides a way to determine the sines contained into a signal.
- The FFT is applied to N successive samples of a signal $x(t)$.
- Let T be the time interval between two samples.
- The *sampling frequency* is $f_s = \frac{1}{T}$.
- The FFT produces N complex numbers $X(k)$, for $k = 0, 1, \dots, N-1$.



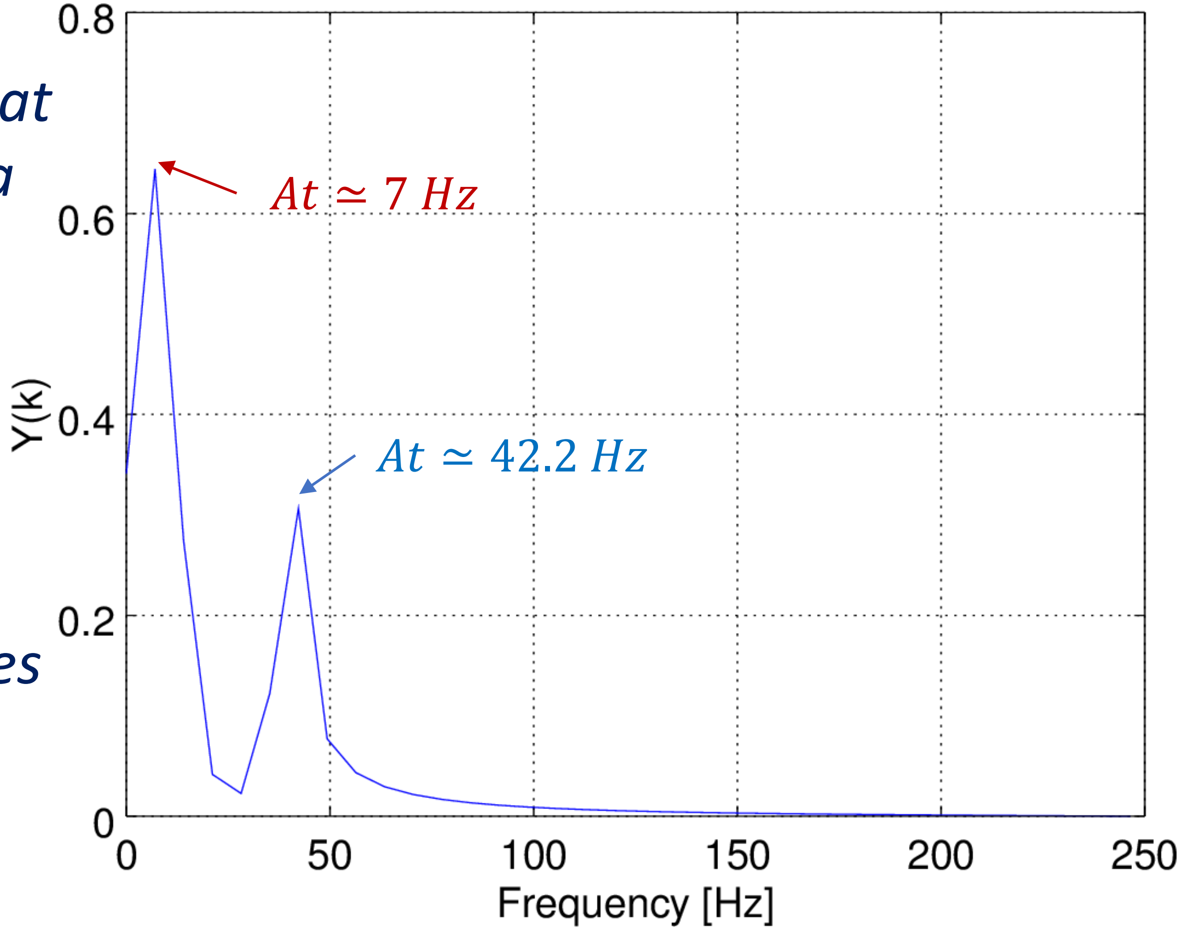
Example

- *The figure shows* $x(t) = \cos(\omega_1 t) + 0.5\cos(\omega_2 t)$ with $f_1 = 10\ Hz$ and $f_2 = 40\ Hz$.
- $N = 71$ *samples are taken over a time interval of* $142\ ms$.
- *The sampling period is* $T = 2\ ms$.
- *The sampling frequency is* $f_s = \frac{1}{T} = 500\ Hz$.



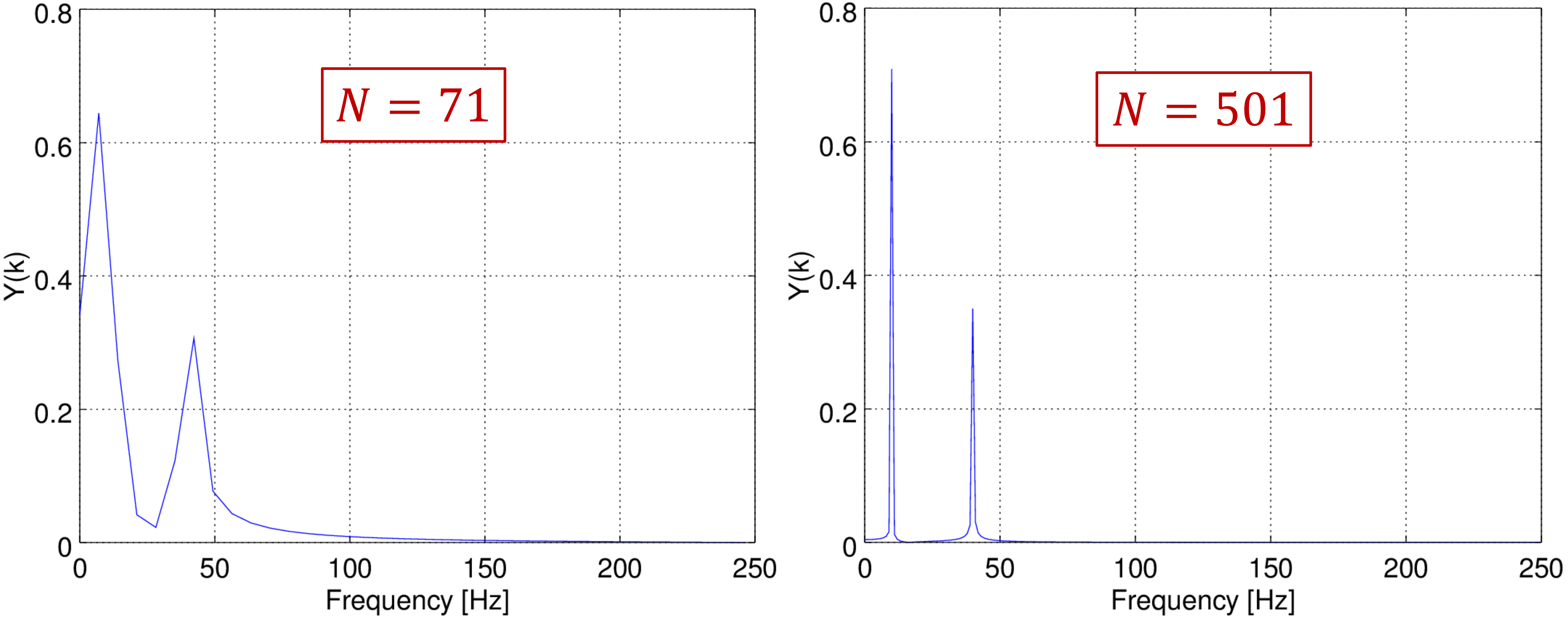
FFT

- The graph $|X(k)|$ vs $k \cdot \frac{f_s}{N}$ for $k = 0, 1, \dots, \left\lceil \frac{N}{2} \right\rceil$ reveals the *frequency spectrum* of $x(t)$.
- The horizontal position of the peaks corresponds to the frequencies of the sines that make up $x(t)$.
- *In our example, the FFT graph indicates that $x(t)$ consists of a sine of about $7\ Hz$ and a sine of about $42.2\ Hz$.*
- *The uncertainty is $\pm \frac{f_s}{2N} = 3.52\ Hz$.*
- *The actual signal is*
  $$x(t) = \cos(\omega_1 t) + 0.5\cos(\omega_2 t)$$
  with $f_1 = 10\ Hz$ and $f_2 = 40\ Hz$.
- *By using a large enough number of samples N, measurements can be made very precise!*



FFT

- *By using a large enough number of samples N, measurements can be made very precise!*

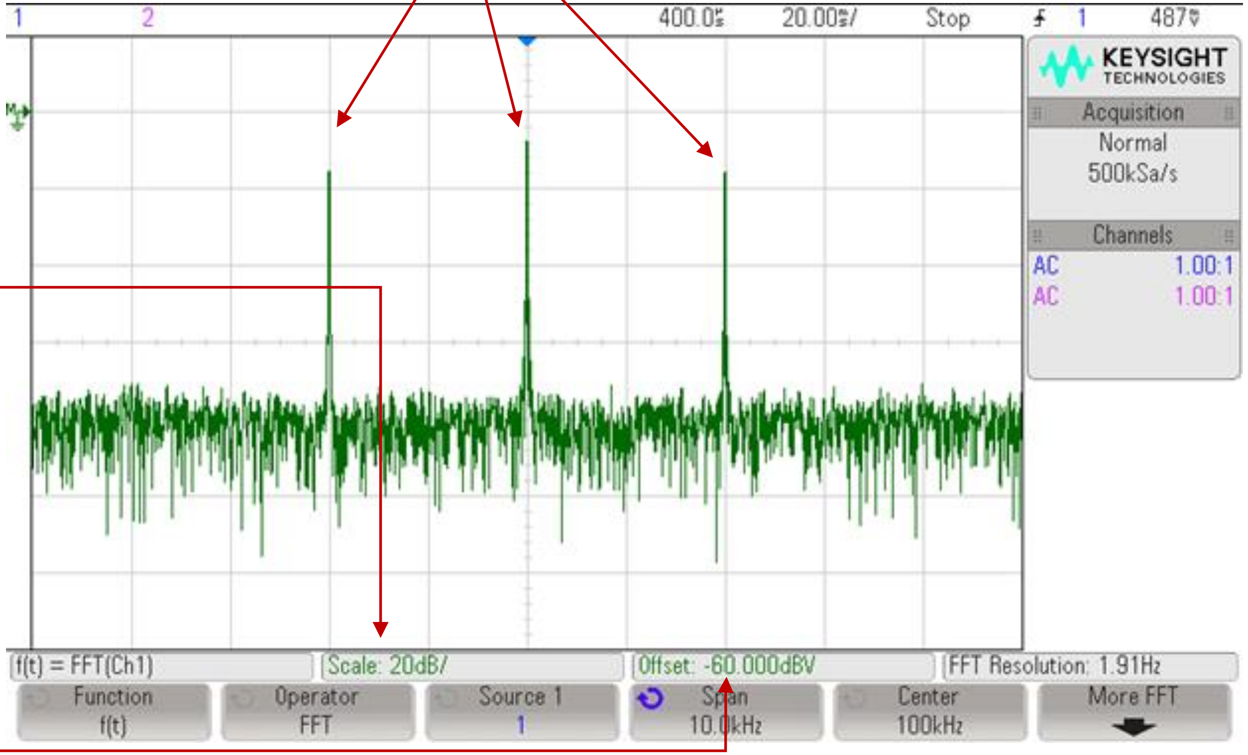


FFT

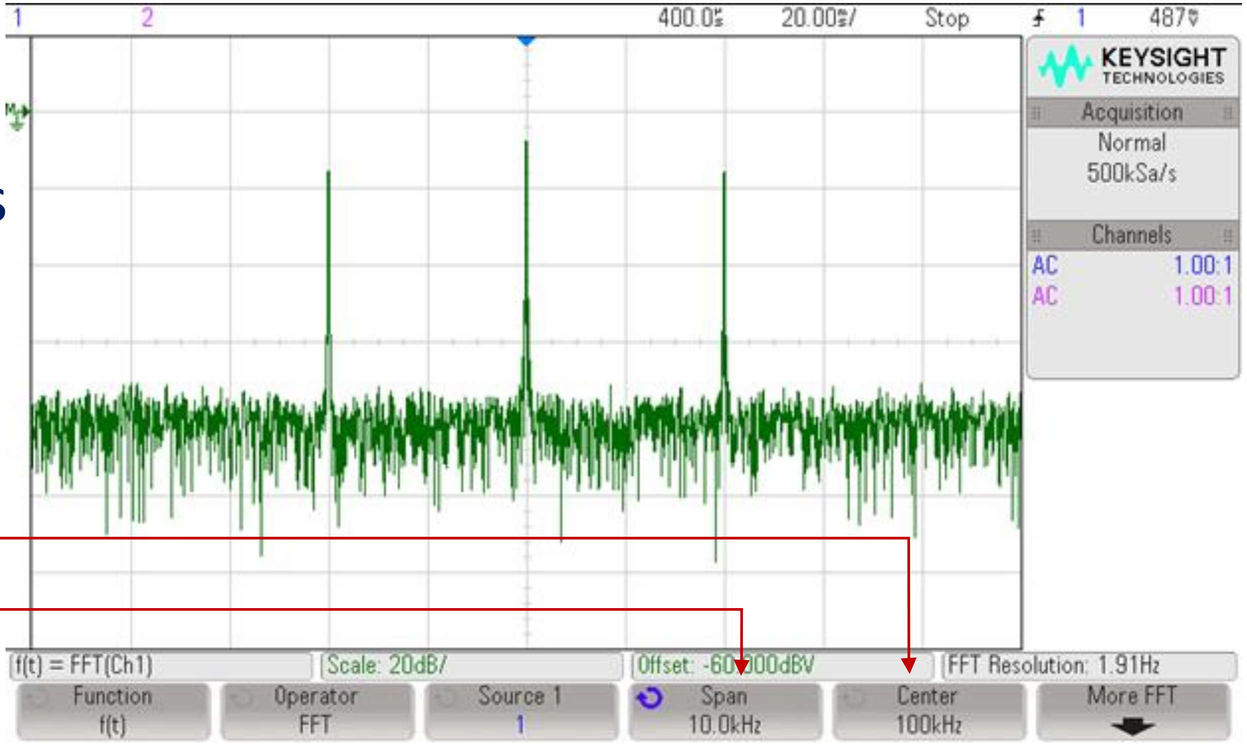
- Even with a large number of samples, a small spectral component located close to a large component might not be visible.
- To address this issue, the samples could be multiplied with the weights of a *windowing function*.
  - This reduces the base of the peaks, making it narrower.
- Windowing functions available on lab oscilloscopes include:
  - Rectangular
  - Flat-top
  - Hamming
  - Blackman-Harris
- The *rectangular window* has all weights equal to one.
  - It does not change the samples (it is as if no windowing function were used).

Example

- The graph shows the FFT plot of a signal consisting of three sines.
- Plots are usually scaled so that the vertical axis is in rms volts or dBV.
- Note the zero level.
- The vertical scale is in dB.
- The zero level is 3 divisions above the middle of the screen.
- With 20 dB/div, the middle of the screen is at $0\ dB - 3\ div \cdot 20\ dB/div$, that is, at $-60\ dB$.
- The rightmost peak is at $0.8\ div$ below the zero level.
- This indicates a sine of $-0.8\ div \cdot 20\frac{dB}{div} = -16\ dBV$, that is, $158.5\ mV$ rms.


f(t) = FFT(Ch1)
Scale: 20dB/
Offset: -60.000dBV
FFT Resolution: 1.91Hz
Function f(t)
Operator FFT
Source 1 1
Span 10.0kHz
Center 100kHz

Example (continued)



f(t) = FFT(Ch1)

Scale: 20dB/

Offset: -60.000dBV

FFT Resolution: 1.91Hz

Function f(t)

Operator FFT

Source 1 1

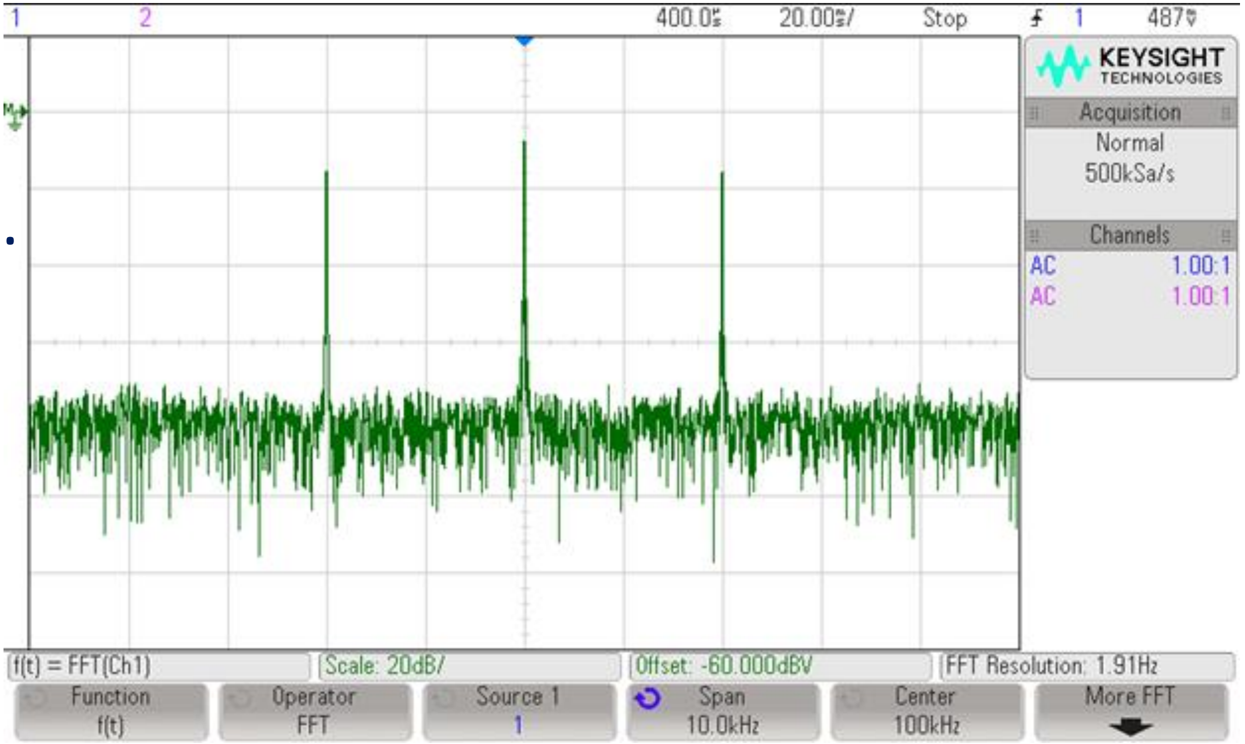
Span 10.0kHz

Center 100kHz

- Similarly, the value of the center peak is $-0.4\ div$, that is, $-8\ dBV$, corresponding to $398.1\ mV$ rms.
- On the horizontal axis, the graph has $100\ kHz$ at the center.
- The graph has a span of $10\ kHz$.
- Since the span has $10\ div$, we have $1\ kHz/div$.
- So the center peak is at $100\ kHz$.
- The right peak is at $100\ kHz + 2\ div \cdot 1 \frac{kHz}{div} = 102\ kHz$.
- The left peak is at $100\ kHz - 2\ div \cdot 1 \frac{kHz}{div} = 98\ kHz$.

Example (continued)

- We conclude that the graph shows a signal consisting of a sine of $98\ kHz$, a sine of $100\ kHz$, and a sine of $102\ kHz$.
- The amplitudes of the sines can be calculated from the graph.
- The graph does not say anything about the phase angles of the sines.



Acquisition Normal 500kSa/s

f(t) = FFT(Ch1)

Scale: 20dB/

Offset: -60.000dBV

FFT Resolution: 1.91Hz

Span 10.0kHz

Center 100kHz